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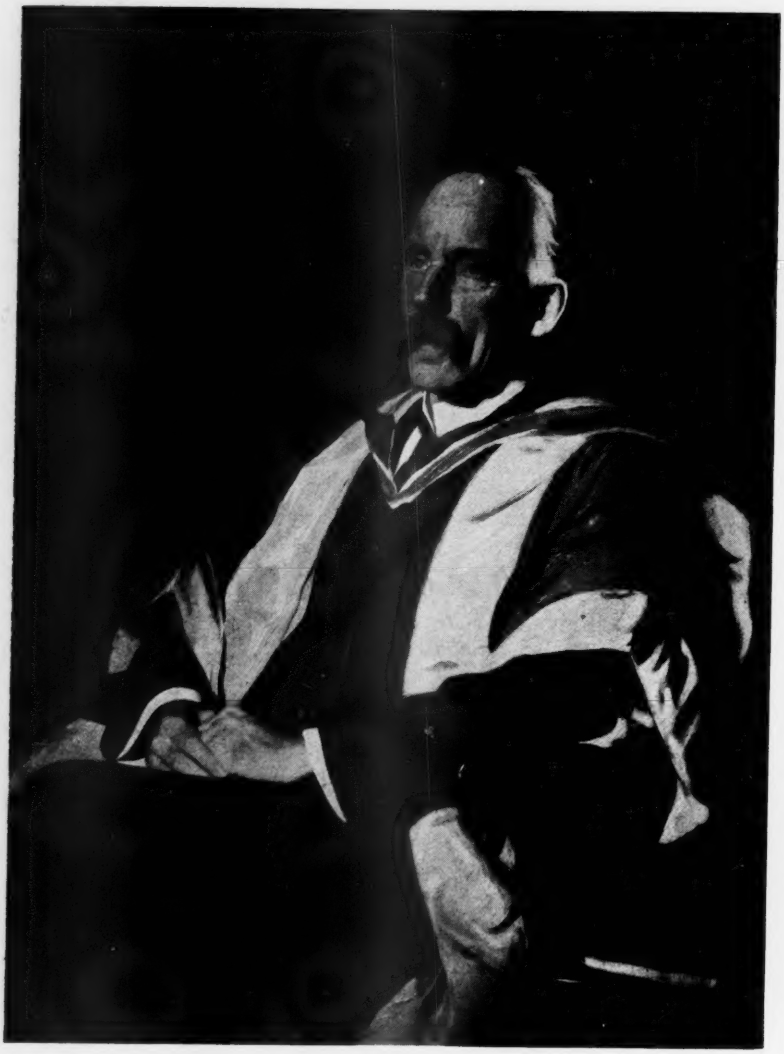

FRANK MORLEY
1860-1937

EDITOR

1900–1937

ON CERTAIN POINTS IN THE THEORY OF ALGEBRAIC DIFFERENTIAL EQUATIONS.*

By J. F. Ritt.

The topics considered in this paper are:

I. *Forms in several unknowns.*
II. *Pairs of forms.*
III. *Essential manifolds composed of one solution.*
IV. *An approximation theorem.*
V. *Essential irreducible manifolds in the manifold of a form.*
VI. *Equations in two unknowns, of the first order.*

The results, under each topic, are described at the head of the section dealing with that topic.

I. Forms in Several Unknowns.

1. Let us consider an algebraically irreducible form A in the unknowns $y_1, \cdots, y_n$. The general solution of A has $n-1$ arbitrary unknowns.[1] It is natural to inquire as to the possibilities for the number of arbitrary unknowns in the other essential irreducible manifolds in the manifold of A. This inquiry is answered by the following theorem:

Given a non-zero form[2] *A in $y_1, \cdots, y_n$, every essential irreducible manifold in the manifold of A has $n-1$ arbitrary unknowns.*

In other words, *every essential irreducible manifold in the manifold of A is the general solution of a form in $y_1, \cdots, y_n$.*

2. It will evidently suffice to prove that every solution

$$y_i = \eta_i, \qquad (i = 1, \cdots, n,) \tag{1}$$

of A is contained in an irreducible manifold, held by A, which has $n-1$ arbitrary unknowns. We wish to show that we may restrict our examination to the case of $\eta_i = 0$, $i = 1, \cdots, n$. Let the η_i in a given solution (1) be

* Received July 27, 1937.

[1] A set of arbitrary unknowns of a system of forms will be called a set of arbitrary unknowns for the manifold of the system.

[2] Algebraic irreducibility is not necessary.

adjoined to the underlying field.[3] Under the substitution $y_i = z_i + \eta_i$, A goes over into a form A' in $z_1, \cdots, z_n$. To prove that $z_i = 0$, $i = 1, \cdots, n$, belongs to an irreducible manifold with $n-1$ arbitrary unknowns, held by A', will be to show that (1) is embedded as described above. Accordingly we assume in what follows that A has the solution $y_i = 0$, $i = 1, \cdots, n$, and we limit ourselves to the study of that solution.

3. Let B represent the sum of the terms of lowest degree in A considered as a polynomial in the y_{ij}. Changing the notation if necessary, we assume that B effectively involves one or more y_{1j}.

Let $v_2, \cdots, v_n$ be functions of x with a common domain of analyticity which, when substituted respectively for $y_2, \cdots, y_n$ in B, reduce B to a form C in y_1 which involves one or more y_{1j} effectively.

Representing by c an arbitrary constant, we put, in A,

$$y_i = v_i c, \qquad (i = 2, \cdots, n.) \tag{2}$$

Then A goes over into an expression D in x, c, y_1.

We shall prove the existence of a formal power series

$$\phi_1 c + \phi_2 c^{p_2} + \cdots + \phi_k c^{p_k} + \cdots, \tag{3}$$

described as in § 11 of our paper, *On the singular solutions of algebraic differential equations*,[4] which causes D to vanish identically in x and c when substituted for y_1 in D.

4. For ϕ_1, we take any function which annuls C above when substituted for y_1 in C. If D vanishes for $y_1 = \phi_1 c$, $\phi_1 c$ can be used for (3). In what follows, we assume that such vanishing does not occur.

5. In D, we put $y_1 = \phi_1 c + u_1$ where u_1 is a new unknown. Then D goes over into an expression H' in x, c, u_1. Designating by m the order of D in y_1, we arrange H' as a polynomial in $u_{10}, \cdots, u_{1m}$. We write

$$H' = a'(c) + \Sigma b'_i(c) u_{10}^{a_{0i}} \cdots u_{1m}^{a_{mi}}. \tag{4}$$

Here $a'(c)$ and the $b'(c)$ are polynomials in c with coefficients which are functions of x. The terms in Σ are those which are not free of $u_{10}, \cdots, u_{1m}$ and we understand that no b'_i is zero. We know that $a'(c)$, which equals $D(\phi_1 c)$, is not zero. As to i, it ranges from unity to some positive integer.

---

[3] If necessary, with a shrinking of the domain of meromorphicity for the field.

[4] *Annals of Mathematics*, vol. 37 (1936), p. 552. Denoted below by *S. S.*

Let σ' be the least exponent of c in a' and σ'_i the least exponent of c in $b'_i(c)$. Let

$$\rho_2 = \text{Max}\, \frac{\sigma' - \sigma'_i}{\alpha_{0i} + \cdots + \alpha_{mi}} \tag{5}$$

where i has the range which it has in Σ.

We shall prove that $\rho_2 > 1$.

Let r be the degree of B of § 3. Let us consider D as a polynomial in c and the y_{1j}. Then each term of D has a degree not less than r. There are terms of degree r, and their sum is obtained by making the substitution (2) in B. Because $C(\phi_1) = 0$, the sum just mentioned vanishes for $y_1 = \phi_1 c$. Thus, as $a'(c) = D(\phi_1 c)$, we have $\sigma' > r$.

Under the substitution $y_1 = \phi_1 c + u_1$, a term of D of degree s in c and the y_{1j} produces a set of terms of degree s in c and the u_{1j}. In particular, the terms of degree r in D will contribute to H' power products in c and the u_{1j} effectively involving the u_{1j} and of degree r. This means that, for certain i, we have, in (4),

$$\sigma'_i + \alpha_{0i} + \cdots + \alpha_{mi} = r.$$

For such i, we have, since $\sigma' > r$,

$$\sigma' - \sigma'_i > \alpha_{0i} + \cdots + \alpha_{mi}.$$

This, by (5), shows that $\rho_2 > 1$.

6. We now take over §§ 12-15 of *S. S.*, substituting y_1 for u_0 and the expression "solution of D of type (3)" for the expression "solution of H of type (15)." Using the argument of § 16 of *S. S.*, we have the series (3) sought for D.

7. Let us suppose now that the solution $y_i = 0$, $i = 1, \cdots, n$, of A, is not contained in a manifold with $n - 1$ arbitrary unknowns. In a decomposition of A into closed essential irreducible systems, let $\Sigma_1, \cdots, \Sigma_t$ be those systems which admit the solution $y_i = 0$, $i = 1, \cdots, n$. Let A_i be a non-zero form in Σ_i, $i = 1, \cdots, t$, involving only $y_2, \cdots, y_n$. Let $E = A_1 A_2 \cdots A_t$.

We choose v_i as in § 3 which do not annul the sum of the terms of lowest degree in E. Then E does not vanish identically in x and c for $y_i = v_i c$, $i = 2, \cdots, n$. Thus, in the decomposition of A, there is some essential irreducible system distinct from $\Sigma_1, \cdots, \Sigma_t$ whose forms all vanish for $y_i = v_i c$, $i = 2, \cdots, n$ and for y_1 as in (3).[5] Such a system must admit the solution $y_i = 0$, $i = 1, \cdots, n$. This completes the proof of the theorem stated in § 1.

[5] Cf. *S. S.*, § 17. Note that A vanishes for the indicated substitutions.

II. Pairs of Forms.

8. We prove the following theorem. *Let A and B be non-zero forms in $y_1, \cdots, y_n$. Let B hold A. Let A_1 be the sum of the terms of lowest degree in A considered as a polynomial in the y_{ij} and let B_1 be the corresponding sum for B. Then B_1 holds A_1.*

A similar result holds for the terms of highest degree.

9. *Remark.* The simplest case is that in which B is a linear combination of A and of the derivatives of A. One might expect that B_1 would then be a linear combination of A_1 and of its derivatives. We shall show by means of an example that this need not be so.

Let

$$A = y_1^2 + y^3; \quad B = 2y_2 A - y_1 \frac{dA}{dx} = 2y^3 y_2 - 3y^2 y_1^2$$

be forms in the unknown y. Then $A_1 = y_1^2$, $B_1 = B$. If B_1 were a linear combination of A_1 and its derivatives, $y^3 y_2$ would be such a linear combination. If the weight of y_j is defined as j, $y^3 y_2$ and y_1^2 have weight 2 and the derivatives of y_1^2 have weight in excess of 2. Thus, $y^3 y_2$ would have to be simply a multiple of y_1^2. This proves our statement. From the expression of B in terms of A, one might now conjecture that *some power* of B_1 is a linear combination of A_1 and of the *first derivative of* A_1. In that case, some power of $y^3 y_2$ would be such a linear combination. This is impossible, since $y^3 y_2$ is not divisible by y_1. Actually, the cube of B_1 is linear in A_1 and its first two derivatives.

10. We enter into our proof. If A_1 is free of the unknowns, B_1 certainly holds A_1. In what follows, we assume that the terms of A_1 are of positive degree. Then A vanishes for $y_i = 0$, $i = 1, \cdots, n$.

We shall prove the permissibility of assuming that A_1 contains a term involving only the y_{1j}. Let $z_2, \cdots, z_n$ and $w_2, \cdots, w_n$ be new unknowns. Let y_i, for $i > 1$, be replaced in A_1 by $z_i + w_i$. Then A_1 goes over into a form C in y_1, the z_i and w_i. C contains terms free of the z_{ij}; the sum D of such terms in C is found by substituting w_i for y_i in A_1 for $i > 1$. Let t_2 be an integer which exceeds the order of D in y_1. On putting $w_2 = y_{1t_2}$ in D, we convert D into a non-zero form D_1 in $y_1, w_3, \cdots, w_n$. We now replace w_3 in D_1 by y_{1t_3}, where t_3 exceeds the order of D_1 in y_1. Continuing, we find a substitution

$$y_i = z_i + y_{1t_i}, \qquad (i = 2, \cdots, n), \tag{6}$$

which converts A_1 into a form E in y_1 and the z_i, E possessing terms free of the z_{ij}. The terms of E will have the same degree as those of A_1.

The substitution (6) may be applied to A and B and will give a situation in which E takes the place of A_1. This proves the legitimacy of the assumption described above, and, in what follows, A_1 will be understood to have terms involving only the y_{1j}.

11. We now refer to § 3. Let

$$y_1 = \phi_1;\ y_i = v_i, \qquad (i = 2, \cdots, n), \tag{7}$$

be any solution of A_1. By § 3, there is a series (3) such that A vanishes identically in x and c when y_i is replaced by $v_i c$ for $i > 1$ and y_1 is replaced by (3). Some power of B is a linear combination of A and its derivatives. Thus, B must vanish for the above replacements. This means that (7) is a solution of B_1, so that our theorem is proved.

12. The case of the terms of highest degree, mentioned in § 8, is perhaps most conveniently handled as follows. Let A_1 and B_1 represent the sums of the terms of highest degree in A and B respectively. Using unknowns $u; z_1, \cdots, z_n$, we put, in A and B,

$$y_i = \frac{z_i}{u^2}, \qquad (i = 1, \cdots, n.) \tag{8}$$

We have then

$$A = C/u^m, \qquad A_1 = C_1/u^m$$
$$B = D/u^m, \qquad B_1 = D_1/u^m$$

with m a positive integer and C, C_1, D, D_1 forms in u and the z_i. C_1 and D_1 will be the sums of the terms of *least* degree in C and D respectively. Because B holds A, uD certainly holds C. By what precedes, uD_1 holds C_1. Because every solution of A_1 yields solutions of C_1 with $u \neq 0$, B_1 holds A_1.

III. Essential Manifolds Composed of One Solution.

13. We use the unknowns $y_1, \cdots, y_n$. We consider a system Σ composed of n forms

$$y_i^{p_i} + F_i, \qquad (i = 1, \cdots, n), \tag{9}$$

where, for each i, p_i is a positive integer and F_i a form which either is identically zero or else is composed of terms each of which is of total degree greater than p_i in the y_{jk}, $j = 1, \cdots, n$; $k \geqq 0$.

We shall prove that *the solution of Σ given by $y_i = 0$, $i = 1, \cdots, n$, is an essential irreducible manifold in the manifold of Σ.*

14. We assume the statement to be false. Then the mentioned solution is contained in an irreducible manifold which is held by Σ but which, for some i, is not held by y_i. Fixing our ideas, let us suppose that y_1 does not hold this manifold. Then there is a value a of x at which the coefficients in the F_i are analytic such that, for every positive integer m and for every $\epsilon > 0$, Σ has a solution, analytic at a, of the type

$$y_i = \sum_{j=0}^{\infty} b_{ij}(x-a)^j, \qquad (i = 1, \cdots, n), \tag{10}$$

with

$$|b_{ij}| < \epsilon, \qquad (i = 1, \cdots, n;\ j = 0, \cdots, m), \tag{11}$$

and with $b_{10} \neq 0$.

We represent by s and c respectively two positive numbers which will be fixed later. Considering a definite solution (10), which corresponds to given m, ϵ, we put

$$z = \frac{x-a}{c}. \tag{12}$$

Then the y_i in (10) become analytic functions of z for z small. Again, we let

$$w_i(z) = c^{-s} y_i(x), \qquad (i = 1, \cdots, n), \tag{13}$$

with z and x related as in (12).

Each equation $y_i^{p_i} + F_i = 0$ goes over into an equation

$$w_i^{p_i} + \sum_{j=1}^{r} c^{\mu_j s - \nu_j} \alpha_j B_j = 0 \tag{14}$$

where the B are power products in the w_k and their derivatives with respect to z, the μ positive integers and the ν non-negative integers. Each α_j is the coefficient in F_i of the power product which produces B_j and we regard the α_j, for any c, as functions of z. It is unnecessary to express the dependence on i of Σ in (14).

If s is large, then, for every i, the $\mu_j s - \nu_j$ in (14) will all be positive. We fix s at a value large enough for this to be realized. Of course, s is independent of m, ϵ and the solution (10).

We have, by (10),

$$w_i(z) = \sum_{j=0}^{\infty} c^{-s+j} b_{ij} z^j, \qquad (i = 1, \cdots, n). \tag{15}$$

We shall now fix c. Let v be the greatest integer less than s. We choose c in such a way that the greatest of the quantities

$$c^{-s+j} |b_{ij}|, \qquad (i = 1, \cdots, n;\ j = 0, \cdots, v),$$

equals unity. This is possible because $b_{10} \neq 0$. Then, if $m \geqq v$ and if ϵ is small, c will be small.

When $1/m$ and ϵ decrease toward zero, the coefficient of z^j in (15) for a fixed j exceeding v, and for any fixed i, tends toward zero. For, if $j > v$ then $j \geqq s$.

It follows that, by decreasing $1/m$ and ϵ, we can select a sequence of solutions (10) which yields, for every i, a sequence of w_i which tends toward a polynomial which is either identically zero or else of degree v at most. The selection can be made in such a way that, for some i, the w_i converge to a polynomial distinct from zero. Fixing our ideas, we assume that the sequence of w_1 tends toward a polynomial $\gamma(z)$ distinct from zero.

We now consider (14) for $i = 1$. When c is small and the w_i are close to their polynomial limits, the expansion of Σ in (14) in powers of z will begin with a large number of small coefficients. This contradicts the fact that $\gamma^{p_1} \neq 0$, so that our result is established.

IV. An Approximation Theorem.

15. In *A. D. E.*,[6] § 74, we established a result equivalent to the following:

Let Σ be a non-trivial closed irreducible system in $y_1, \cdots, y_n$. Let F be any form which does not hold Σ. Let $\zeta_1, \cdots, \zeta_n$ be any solution of Σ, analytic in some area $\mathfrak{B}$. There exists a set of points, dense in $\mathfrak{B}$, such that, given any point a of the set, any positive integer m and any $\epsilon > 0$, Σ has a solution $\xi_1, \cdots, \xi_n$, analytic at a, which does not annul F at a, such that, for every i, each of the first $m + 1$ coefficients in the Taylor expansion of $\xi_i - \zeta_i$ at a is of modulus less than ϵ.

We are going to derive here the following stronger conclusion from the hypothesis in the above theorem. *There exists a set of points, residual*[7] *in* $\mathfrak{B}$, *such that, given any point a of the set, any two positive integers r and m and any $\epsilon > 0$, Σ has a solution $\xi_1, \cdots, \xi_n$, analytic at a, which does not annul F at a, such that, for every i, the r-th roots of $\xi_i - \zeta_i$ are analytic at a and have Taylor expansions at a in which the first $m + 1$ coefficients are all less than ϵ in modulus.*

16. *Remarks.* The residual set of points a is not offered as one of the attractions of the above result. What is noteworthy is the use of the r-th roots. Residual sets occur in the proof, and nothing is lost in using such a set in the statement of the theorem. We note at this point that, in the approximation

[6] *A. D. E.* will stand for our Colloquium Lectures.

[7] The complement in $\mathfrak{B}$ of the sum of a countable number of sets each of which is nowhere dense on $\mathfrak{B}$.

theorem of *A. D. E.*, the dense set of points a may be replaced immediately by a residual set. In short, for the m and ϵ on page 102 of *A. D. E.*, one may use any point a of the area $\mathfrak{A}'$. Thus, the points a which may not be used for given m, ϵ are nowhere dense in $\mathfrak{A}$; the points a for which some impossible pair m, ϵ can be found form a set of the first category.

The stronger approximation theorem, which, as will be seen in § 26, is not without utility, is a first result in a program to perfect the approximation theorem of *A. D. E.* It would be natural to conjecture that $\zeta_1, \cdots, \zeta_n$ can be embedded analytically in a one-parameter family of solutions which do not annul F. That this is not so can be seen from § 90 of *S. S.*, where the singular solution $y = 0$ of

$$y_2{}^2 - y_1 + y = 0 \tag{16}$$

is discussed. That singular solution belongs to the general solution of (16). If (16) were satisfied, for every small h, by

$$y = \phi(x, h)$$

with ϕ analytic for x in some area and h small, and with ϕ vanishing identically in x for $h = 0$ but not vanishing identically in x and h, the first member of (16) would have a convergent y-solution. However, the only y-solution which exists is divergent for every $y \neq 0$.

Having disposed of the above conjecture, one might ask whether $\zeta_1, \cdots, \zeta_n$ cannot always be approximated uniformly in some area by solutions which do not annul F. Such a result would certainly not imply immediately the one which will be established here. For instance, as h approaches zero, $y = h^2 + hx$ approaches zero uniformly in any bounded area, but the coefficient of x in the expansion of $y^{1/2}$ at $x = 0$ does not tend toward zero with h.

17. We shall show that it will suffice to consider the case in which the ζ_i are identically zero. We adjoin the n functions ζ_i to the underlying field, limiting the domain of x if necessary. Σ becomes equivalent, for the enlarged field, to one or more essential irreducible systems. One of these systems, call it Σ', will admit a sequence of solutions which do not annul F and which approach $\zeta_1, \cdots, \zeta_n$ in the manner described in *A. D. E.*, § 74. Then $\zeta_1, \cdots, \zeta_n$ is a solution of Σ' and F does not hold Σ'. Under the substitution $y_i = z_i + \zeta_i$, Σ' goes over into an irreducible system Σ'' in the z_i; F goes over into a form G in the z_i. Then $z_i = 0$, $i = 1, \cdots, n$ will be a solution of Σ'' and G will not hold Σ''. This is enough for the justification of our statement.

18. Let P be a form in $y_1, \cdots, y_n$, given by

$$P = y_1^p(1 + A) + B \tag{17}$$

where p is a positive integer, A a form vanishing for $y_i = 0$, $i = 1, \cdots, n$, and B a form which, when written as a polynomial in y_1 and its derivatives, has no term of degree less than $p + 1$.

It is easy to prove, subjecting x and y_1 to a Painlevé transformation, that the solution $y_i = 0$, $i = 1, \cdots, n$, of P is not contained in any irreducible manifold which is held by P but not by y_1.

19. In accordance with § 17, we assume that $\zeta_i = 0$, $i = 1, \cdots, n$.

Let y_j be any one of the n unknowns. In the first stage of our proof we shall show that, given any positive integer r, there exists a dense set of values of x such that, given any a of the set and any m and ϵ, Σ has a solution $\xi_1, \cdots, \xi_n$ which does not annul F at a, the r-th roots of ξ_j being analytic at a and the first $m + 1$ coefficients in the expansions at a of

$$\xi_1, \cdots, \xi_{j-1}, \xi_j^{1/r}, \xi_{j+1}, \cdots, \xi_n$$

all having moduli less than ϵ.

If y_j holds Σ, our result is seen directly to hold. In what follows we assume that y_j does not hold Σ.

20. Let $y_1, \cdots, y_q$ be a set of arbitrary unknowns for Σ (if such a set exists). Whether or not y_j is among these arbitrary unknowns is of no importance. Let

$$A_{q+1}, \cdots, A_n \tag{18}$$

be a basic set for Σ, introducing $y_{q+1}, \cdots, y_n$. Let S_i represent the separant of A_i in (18).

No generality is lost in assuming that F of § 15 is divisible by y_j and by each S_i. In what follows, we assume such divisibility.

Let the r of § 19 be given and let an m then be selected. We make the non-restrictive assumption that, for every i, m exceeds the order of F in y_i. We consider the forms in (18) and also the first m derivatives of each of them. We secure thus a set of $(m + 1)(n - q)$ forms which we shall now regard as simple forms in the y_{ik}. The set of simple forms thus obtained will be denoted by Φ and the unknowns in Φ will be taken as those y_{ik} for which $k \leqq r_i + m$ where r_i is the highest of the orders in y_i of the forms in (18).

Let Π be the set of simple forms in the unknowns just described which vanish for all solutions of Φ which annul no S_i. It is easily proved that Π is a prime system. (Cf. *A. D. E.*, § 73.)

Because Σ has the solution $y_i = 0$, Π has a solution with every y_{ik} zero. Also Π is not held by F.

Let $p = mr$. Introducing a new unknown v, we put, in Π, $y_{j0} = v^p$. Then Π goes over into a system Λ in v and the y_{ik} distinct from y_{j0}. Let F with y_{j0} replaced by v^p be represented by C. Then C does not hold Λ.

Let $y_{j1}, \cdots, y_{jt}$ be those unknowns in Λ which arise from derivatives (proper) of y_j. We put, in Λ,

$$y_{ji} = v^{p-1} w_i, \qquad (i = 1, \cdots, t). \tag{19}$$

Then Λ becomes a system Ω in the y_{ik} with $i \neq j$, in v and the w_i. Let D be the form into which C is converted by (19). Then D does not hold Ω.

21. In a decomposition of Ω into essential prime systems, let $\Omega_1, \cdots, \Omega_s$ be those systems which are not held by D. We are going to show that there is some Ω_i each of whose forms vanishes when the unknowns are all replaced by 0.

Let this be false. Then there exists a form K, with no term free of the unknowns, such that $1 + K$ holds every Ω_i. Let g be the degree of K considered as a polynomial in the w_i. We replace each w_i in $1 + K$ by y_{ji}/v^{p-1} and multiply the resulting expression through by $v^{g(p-1)}$. We obtain a form R given by

$$R = v^{g(p-1)} + M$$

where M is a form of the following description. Every term of M which involves no y_{jk} is of the form

$$\alpha v^{g(p-1)} L$$

with α a function of x and L a power product of positive degree. Every term in M involving some y_{jk} effectively is of the type

$$\alpha v^d y_{j1}^{d_1} \cdots y_{jt}^{d_t} L \tag{20}$$

with L a power product free of the y_{jk} and with

$$d = g(p-1) - (d_1 + \cdots + d_t)(p-1).$$

Thus

$$d + p(d_1 + \cdots + d_t) > g(p-1). \tag{21}$$

In other words,

$$R = v^{g(p-1)}(1 + G) + H \tag{22}$$

where G, free of the y_{jk}, vanishes when the unknowns are all replaced by 0 and where the terms of H, all of which involve the y_{jk}, are of the type (20) with (21) holding. Furthermore, CR holds Λ.[8]

[8] Note that C is divisible by v.

Let ω be a primitive p-th root of unity. We replace v in R successively by $\omega^i v$, $i = 1, \cdots, p$. We obtain a set of forms $A_1, \cdots, A_p$ whose product T is a form with coefficients in $\mathfrak{F}$. T will involve v with exponents which are all multiples of p. Also, CT holds Λ and T has an expression

$$v^{g(p-1)p}(1 + P) + Q \tag{23}$$

where P (whose exponents of v are all multiples of p) vanishes when the unknowns are replaced by 0 and where Q has terms of the type (20) with d divisible by p and with

$$d + p(d_1 + \cdots + d_t) > g(p-1)p. \tag{24}$$

In T, we replace v^p by y_{j0} and consider the resulting form U as a differential polynomial. Then U is of the form

$$y_j^{g(p-1)}(1 + V) + W$$

where V vanishes for $y_i = 0$, $i = 1, \cdots, n$ and where the degree of each term of W in y_j and its derivatives exceeds $g(p-1)$.

Now FU holds Σ. Then U holds Σ. According to § 18, the solution $y_i = 0$, $i = 1, \cdots, n$ of U cannot be approximated by solutions of U with $y_j \neq 0$. This contradicts the fact that U holds Σ. We have thus proved the statement made at the head of the present section.

22. Fixing our ideas, let us suppose that every form in Ω_1 vanishes when the unknowns are all replaced by 0.

Considering Ω_1 and D, we refer to § 68 of *S. S.* Corresponding to each unknown in Ω_1, we find a function $\phi(x, h)$, analytic for x in some area and for h small, with $\phi(x, 0)$ equal to 0 for every x. The substitution of all of the $\phi(x, h)$ for their corresponding unknowns produces 0 for each form of Ω_1 but not for D.

Passing from Ω_1 to Λ, we have a $\phi(x, h)$ for every unknown in Λ, with each ϕ identically zero in x for $h = 0$, and with each form in Λ, but not C, reducing to zero for every x and h when the unknowns are replaced by the ϕ. Let the expansion in powers of h of the ϕ for v begin with a term in h^d with $d > 0$.[9] Then the expansion of the ϕ for any y_{jk} will, if it is not identically zero, begin with a power of h whose exponent exceeds $(p-1)d$.

We now consider Π. For the unknowns in Π we have a set of ϕ which annul the forms in Π but not F. The lowest exponent of h in the ϕ for y_{j0}

[9] Because D is divisible by v, the ϕ for v is not identically zero.

will be pd while the lowest exponent for any y_{jk} with $k > 0$ will exceed $(p-1)d$.

In what follows, the ϕ used will be those for Π.

23. We shall use an area $\mathfrak{A}_1$ in the plane of x which satisfies the following conditions:

(a) The coefficients in F and in the A_i in (18) are analytic in $\mathfrak{A}_1$.

(b) The ϕ are analytic for x in $\mathfrak{A}_1$ and h small.

(c) The coefficient of h^{pd} in the ϕ for y_{j0} vanishes nowhere in $\mathfrak{A}_1$.

(d) The coefficient of the lowest power of h in the result obtained by substituting the ϕ into F vanishes nowhere in $\mathfrak{A}_1$.

24. Let a be any value of x in $\mathfrak{A}_1$. We put $x = a$ in the ϕ, whereupon the ϕ go over into functions ψ of h, analytic for h small and vanishing for $h = 0$.

Every A_i in (18) vanishes when x is replaced by a and the other letters by their corresponding ψ. A similar statement holds for the first m derivatives of the A_i. Let us consider $A_{q+1}^{(m+1)}$, the $(m+1)$-st derivative of A_{q+1} in (18). In $A_{q+1}^{(m+1)}$, we replace x by a and every y_{ik} which occurs in Π by its ψ. For every $i \leqq q$, y_{i,r_i+m+1} with r_i as in § 20, may appear in $A_{q+1}^{(m+1)}$. With such letters, whether or not they figure effectively in $A_{q+1}^{(m+1)}$, we associate functions $\psi(h)$ which are identically zero and, where one of the letters figures effectively in $A_{q+1}^{(m+1)}$, we replace it in that form by 0. After these substitutions, $A_{q+1}^{(m+1)}$ becomes a linear expression in

$$y_{q+1,r_{q+1}+m+1} \tag{25}$$

with coefficients which are functions of h. The coefficient of the letter (25) will be S_{q+1} with substitutions as above. Because F is divisible by S_{q+1} and because (d) of § 23 holds, that coefficient is not zero. When the expression obtained from $A_{q+1}^{(m+1)}$ is equated to zero, the letter (25) is determined as a function of h which is analytic, or has a pole, for $h = 0$. We treat $A_{q+2}^{(m+1)}$ similarly, substituting for the letter (25) the function of h just found. We proceed similarly with the $(m+1)$-st derivatives of the other A_i. When this step is concluded we treat in a similar way the higher derivatives of the A_i.

The net result of the total operation is as follows. We obtain a set of functions ψ of h, one ψ for every y_{ik}. The ψ are analytic, or have a pole, for $h = 0$; in particular the ψ for $k \leqq m$ are all analytic, and equal to zero, for $h = 0$. For $i \leqq q$, and for $k > m$, the ψ are zero. The ψ for y_{j0} has a least exponent of h equal to pd, while, for a y_{jk} with $0 < k \leqq m$, and with a ψ not identically zero, the least exponent exceeds $(p-1)d$. There is a $\delta > 0$ such that the ψ are all analytic for $|h| < \delta$, except, perhaps, for $h = 0$.

For h small and distinct from zero, the ψ become numbers which are derivatives in a normal solution of (18), the solution being analytic for $x = a$ and not annulling F at a. In such a solution, the y_i with $i \leqq q$ will be polynomials.

We examine y_j in these solutions. There is furnished for it, by what precedes, an expansion

$$\psi_0(h) + \psi_1(h)(x-a) + \cdots + \psi_k(h)\frac{(x-a)^k}{k!} + \cdots, \tag{26}$$

where the lowest exponent of h in ψ_0 is pd, while that in ψ_k with $k \leqq m$ exceeds $(p-1)d$.

For h small, and not zero, the r-th roots of y_j are analytic at a.

We write the series (26) in the form

$$\psi_0(h)[1 + \beta_1(h)(x-a) + \cdots]. \tag{27}$$

Then, for $k \leqq m$, the expansion about $h = 0$ of β_k is either zero or has a least exponent which exceeds $-d$.

Since $p = mr$, the least exponent of h in ψ_0 is mrd. Then the r-th roots of ψ_0 are analytic at the origin. Let $\gamma(h)$ be such an r-th root. The least exponent of h in γ is md.

One of the r-th roots of the bracket in (27) has an expansion

$$1 + \delta_1(h)(x-a) + \cdots$$

in which δ_k with $k \leqq m$ begins with a term in h of exponent greater than $-kd$.

Thus, in an r-th root of y_j as given by (26), the coefficient of $(x-a)^k$ with $k \leqq m$ will begin with a positive power of h. This is enough to prove the result stated in § 19.[10]

25. We conclude the proof of the result stated in § 15. Consider any r. Under the substitution $y_1 = v_1{}^r$, Σ goes over into a system Λ and F into a form F_1, in $v, y_2, \cdots, y_n$. From what goes before, it follows that some essential irreducible system Σ_1 in the decomposition of Λ contains the solution $v_1 = 0$; $y_i = 0$, $i > 1$, and is not held by F_1. We give Σ_1, with respect to y_2, the treatment accorded to Σ with respect to y_1. Continuing, we reach a system Σ_n and a form F_n in unknowns $v_1, \cdots, v_n$, such that F_n does not hold Σ_n and that $v_i = 0$, $i = 1, \cdots, n$, is a solution of Σ_n. This solution of Σ_n can be approximated at the points a of a residual set by solutions which do not annul F_n at a and which have expansions at a with as many small coefficients

[10] For m as given, any a in $\mathfrak{A}_1$ will serve for every ϵ. Thus there is a dense set of points a, in fact a residual set, which serve for all m, ϵ.

as one pleases. For the particular r used, the transformation $y_i = v_i^r$ gives the solutions $\xi_1, \cdots, \xi_n$ of § 15. Furthermore, the residual sets which correspond to the values 1, 2, 3, $\cdots$, of r have a residual set in common. The result of § 15 is thus completely proved.

26. As an application, we consider a form

$$G = y_1^{p_1} \cdots y_n^{p_n} + A$$

where the p_i are non-negative integers whose sum is positive and where A is a form of the following description:

(a) Each term of A has a degree in the y_{ik} which exceeds $p_1 + \cdots + p_n$.

(b) Given any term L of A, and any y_j, L is either divisible by $y_j^{p_j}$ or else of degree higher than p_j in the y_{jk}.

We shall prove that the *solution* $y_i = 0$, $i = 1, \cdots, n$ *of* G *is not contained in any irreducible manifold which is held by* G *but not by* $y_1 y_2 \cdots y_n$.

This result appears not to be obtainable readily through the Painlevé transformation.

Using a positive integer r which will be fixed in a moment, we replace each y_i in G by v_i^r. If r is large enough, A will go over into a form in which each term is divisible by $v_1^{p_1 r} \cdots v_n^{p_n r}$. Let r be thus taken. Then G goes over into a form H which is the product of $v_1^{p_1 r} \cdots v_n^{p_n r}$ by a form K of the type

$$1 + B$$

where B vanishes for $v_i = 0$, $i = 1, \cdots, n$.

If our result were not true, it would follow from § 15 that K has solutions which, for suitable points a, have expansions which begin with as many arbitrarily small coefficients as one may desire. Q. E. D.

V. Essential Irreducible Manifolds in the Manifold of a Form.

27. With a view towards later applications, we shall extend here the results of *S. S.*, Part I, to forms in several unknowns.

28. Following *S. S.*, §§ 1-3, one secures the following result.

Let F and A be two forms in $y_1, \cdots, y_n$, both of class n and algebraically irreducible. Let the orders of F and A in y_n be m and $l < m$ respectively. Let A_j represent the j-th derivative of A and S the separant of A. Then *there exist a non-negative integer* t *and a positive integer* r *such that* $S^t F$ *has a representation*

$$\sum_{j=1}^{r} C_j A^{p_j} A_1^{i_{1j}} A_2^{i_{2j}} \cdots A_{m-l}^{i_{m-l,j}}, \qquad (28)$$

with non-negative p_j *and* i_{kj}, *where no two of the r sets* $i_{1j}, \cdots, i_{m-l,j}$ *are identical; the* C_j *being forms which are of orders not exceeding* l *in* y_n *and which are not divisible by* A.

For any admissible t, (28) is unique. In what follows, the smallest t will be used.

29. Let F hold the general solution of A. Then, *for the general solution of* A *to be an essential irreducible manifold in the manifold of* F, *it is necessary and sufficient that* (28) *possess a term of the type* $C_j A^{p_j}$, *which term, if* (28) *is considered as a polynomial in* $A, A_1, \cdots, A_{m-l}$, *is of lower degree than every other term of* (28).

The sufficiency proof proceeds as in § 6 of *S. S.*

For the necessity proof, we assume that, among the terms of lowest degree in (28), there is a term which involves derivatives of A. We prove that the general solution of A is not essential.

According to Part I of the present paper, the manifold of F consists of the general solutions of certain forms $B_1, \cdots, B_d$. If there are B_i whose orders in y_n do not exceed l, let T denote the product of such B_i. Otherwise, let $T = 1$. Let T be arranged as a polynomial in the y_{nj} and let U be any coefficient in T. Then U, being a form in $y_1, \cdots, y_{n-1}$, does not hold the general solution of A.

Considering (28) as a polynomial in A and the A_j, we take its terms of lowest degree and select from them those terms which have a highest degree in A_{m-l}. From the terms just taken, we select those for which the degree in A_{m-l-1} is highest. We continue through A_1. Our process isolates a single term of (28), with a definite C_j. This C_j will be used in what follows.

We put $E = UC_jS$. Let $\bar{y}_1, \cdots, \bar{y}_n$ be any solution in the general solution of A which does not annul E. We put, in (28),

$$y_i = \bar{y}_i, \; i = 1, \cdots, n-1; \; y_n = \bar{y}_n + u_0, \tag{29}$$

with u_0 a new unknown. By *S. S.*, § 10, (28) goes over into a form in u_0 which vanishes for $u_0 = 0$ and which has, among its terms of lowest degree in the u_{0i}, terms of order higher than l in u_0.

According to §§ 11-16 of *S. S.*, F has a formal solution

$$y_i = \bar{y}_i, i < n; \; y_n = \bar{y}_n + \phi_1 c + \cdots \tag{30}$$

where ϕ_1 is any solution of a differential equation of order higher than l.

We follow § 17 of *S. S.* Let Σ_i, $i = 1, \cdots, d$, be the closed system whose manifold is the general solution of B_i above. Then every solution (30) is a

formal solution of some Σ_i. We say that there is a solution (30) which is a solution of some Σ_i whose B_i has an order higher than l in y_n. Let this be false. Then every solution (30) annuls T above. Under the substitution (29), T goes over into a form $V(u_0)$ which is annulled by every series $\phi_1 c + \cdots$. Because the $\bar{y}_i$ with $i < n$ do not annul U, V is not identically zero. The order of V in u_0 does not exceed l. The proof is now completed as in *S. S.*, § 17.

VI. Equations in Two Unknowns, of the First Order.

Generalities.

30. We deal with an algebraically irreducible form F in the unknowns u and v. F has an order in u and an order in v. We shall assume that the maximum of these two orders is unity.

The manifold of F consists of the general solutions of forms

$$F, B_1, \cdots, B_s. \tag{31}$$

The B_i are determined by the methods of *A. D. E.*, Chapter V, with the help of Part V of the present paper.

Clearing the ground for further operations, we shall show that, if there are B_i in (31), they are of order zero in u and in v. Let us consider B_1. Because F holds the general solution of B_1, the remainder of F with respect to B_1, say for the order u, v of the unknowns, is zero. Thus, the order of B_1 in v cannot exceed unity. If that order were unity, F would be divisible by B_1. Thus, the B_i are simple forms.

31. The problem to which this Part VI is devoted is that of determining the solutions which any B_i, call it B, has in common with the general solution of F.

Fixing our ideas, we assume that B is not free of v. Following § 28, we write

$$S^t F = C_0 B^p + C_1 B^{p_1} B'^{q_1} + \cdots + C_r B^{p_r} B'^{q_r}$$

with S and B' the separant and derivative respectively of B. The orders of the C in v and in u do not exceed 0 and 1 respectively and no C is divisible by B. For every i, $p < p_i + q_i$.

The sufficiency proof of the result in § 29 brings out the fact that every solution which B has in common with the general solution of F is a solution of C_0. Let us suppose that B and C_0 have common solutions. Using the order u, v of the unknowns, let basic sets be obtained for a set of closed irreducible systems whose manifolds make up the manifold of the system C_0, B.

We shall prove that each basic set consists of two forms. The resultant of C_0 and B with respect to v is a non-zero form in u alone, of order at most unity in u. Hence, given any closed irreducible system Σ held by C_0 and B, a basic set of Σ starts with a form U in u, of order at most unity. Because B involves v, B is not divisible by U. Hence the remainder of B with respect to U is not zero. Thus, the basic set of Σ has a second form, of order zero in v, which introduces v.

Let then

$$U, V \tag{32}$$

be a basic set for Σ, one of the closed irreducible systems considered above. U is of order at most unity in u and V is of order zero in v.

It will be proved in § 68 that, *if U is of order unity, the manifold of Σ is contained in the general solution of F.*

As to the case in which U is of order zero, a theorem of E. Gourin[11] shows that if Σ has a solution in common with the general solution of F, the manifold of Σ is contained in that general solution. It thus becomes a question of deciding whether a given solution, $u=\eta$, $v=\zeta$ of Σ is contained in the general solution. As in *S. S.*, § 88, this case can be reduced to the case of $u=v=0$.

We shall therefore undertake the investigation of the following problem. *Let F vanish for $u=v=0$. It is required to determine whether $u=v=0$ is contained in the general solution of F.*

The case of interest, of course, is that in which one or more B_i in (31) vanish for $u=v=0$.

Through § 67, in which a summary of results is given, we shall be occupied with the problem just stated. Thus, through § 67, F *will vanish for* $u=v=0$.

Elements.

32. We consider a relation of the type

$$v=\phi_1 u^{\rho_1}+\cdots+\phi_k u^{\rho_k}+\cdots. \tag{33}$$

The ρ are positive rational numbers with a common denominator, which increase with their subscripts. The ϕ are functions of x, all analytic in some area.[12] It is understood that the second member of (33) may be identically zero. If we differentiate (33) formally, we secure a relation

[11] *Bulletin of the American Mathematical Society*, vol. 39 (1933), p. 593.

[12] In the definition of y-solution in *S. S.*, only a common point of analyticity was demanded of the coefficients. The reason for using an area here will appear in § 55.

(34) $$v_1 = \frac{\partial v}{\partial x} + \frac{\partial v}{\partial u} u_1.$$

The series for $\partial v/\partial u$ may contain a finite number of negative powers of u.

The substitution of the above expressions for v and v_1 into F produces a polynomial in u_1 whose coefficients are series in u which may contain a finite number of negative powers. If this polynomial in u_1 vanishes identically, we shall call the second member of (33) an *element* of F.

For instance, if $F = v_1 + v - u_1 - u$, F has u as an element. Examples of elements can be given, with constant ϕ, which diverge for every $u \neq 0$.

Multiplicities.

33. Let an element of F be given by (33). If

(35) $$\frac{\partial F}{\partial v}, \cdots, \frac{\partial^{p-1} F}{\partial v^{p-1}}$$

all vanish identically in x, u, u_1 when v is taken as in (33), while $\partial^p F/\partial v^p$ does not, the element will be said to be of *multiplicity* p.

If F has an element, v or v_1 must figure in F. We shall prove that an element of F has a multiplicity if and only if F, considered as a polynomial in v, is of positive degree.

If F is of zero degree in v, every $\partial^i F/\partial v^i$ vanishes identically. This proves the necessity of the condition. Let the condition be satisfied and let

$$F = \alpha_0 + \alpha_1 v + \cdots + \alpha_n v^n$$

with $n \geqq 1$. Suppose that the n forms $\partial^i F/\partial v^i$, $i = 1, \cdots, n$, all vanish for (33). We have

$$\frac{\partial^n F}{\partial v^n} = n\,!\,\alpha_n,$$

so that α_n vanishes for (33). Again,

$$\frac{\partial^{n-1} F}{\partial v^{n-1}} = (n-1)\,!\,\alpha_{n-1} + n\,!\,\alpha_n v$$

so that α_{n-1} must vanish for (33). Continuing, we find every α_i to vanish for (33). Because F is algebraically irreducible, $\alpha_0, \cdots, \alpha_n$ are relatively prime as polynomials in u, u_1, v_1. Hence, some linear combination of them, with suitable forms for coefficients, is a non-zero form free of v_1, that is, a non-zero form in u. Such a form cannot vanish for (33). This proves the sufficiency.

Strong Sums.

34. We consider the effect of making, in F, the substitution

$$v = \phi_1 u^{\rho_1} + \cdots + \phi_k u^{\rho_k} \tag{36}$$

where the k numbers ρ are positive and rational, and increase with their subscripts. It is understood that the second member of (36) may be identically zero.

Under this substitution, F goes over into a finite sum

$$\Sigma a_i u^{\alpha_i} u_1^{\beta_i} \tag{37}$$

with each a_i a function of x, each α_i a rational number (possibly negative), and each β_i a non-negative integer.

If (37) is not identically zero, and if, among those of its terms for which $\alpha_i + \beta_i$ is a minimum, there are terms with $\beta_i > 0$, we shall call the second member of (36) a *strong sum* for F.

Example 1. $v^2 - uv + v_1^4 + u_1^3$ has u as a strong sum. In fact, $u + u^\rho$, with ρ any sufficiently large rational number, will be a strong sum.

Example 2. $u + u_1 v_1$ has no strong sum.

From any strong sum of a form F, new strong sums can be derived, as in Example 1, by the addition of terms.

The part played by elements and strong sums will be as follows. Suppose that the manifold of the form u is not contained in the general solution of F. It will turn out that for $u = v = 0$ to be in the general solution, it is necessary and sufficient that F have either a strong sum, or else an element which causes no B_i in (31) to vanish when substituted for v.

Indices.

35. F is to be as in § 30 and is to admit $u = v = 0$ as a solution. We denote by $\mathfrak{A}$ the region in which the coefficients in F are meromorphic.

Suppose first that F has no strong sum. We shall call a positive integer n the *index* of F if there exists a set of points $\mathcal{E}$, contained in $\mathfrak{A}$ and having no limit point in $\mathfrak{A}$, such that, given any simply connected region $\mathfrak{A}_1$ in $\mathfrak{A}$ which contains no point of $\mathcal{E}$, F has a finite set of distinct elements which satisfy the following conditions:

(a) *The coefficients in the elements are analytic throughout* $\mathfrak{A}_1$.

(b) *The elements have multiplicities and the sum of the multiplicities for the elements of the set is* n.

(c) *Every element of F with coefficients analytic in some area contained in $\mathfrak{A}_1$ coincides with some element of the set.*

It is easy to see that there can be no more than one n as above.

If F has no strong sum and no elements, the index of F will be defined as zero.

If F has a strong sum, the index of F will be defined as ∞.

Our work will show that if F has no strong sum, but has elements, a positive n exists as above. Thus an index will be known to exist for every F which satisfies our assumptions. The index will play a rôle analogous to that of the *y-solution number* in *S. S.*

POLYGONS.

36. We write F in the form

$$\sum_{i=1}^{r} a_i u^{\alpha_i} u_1^{\beta_i} v^{\gamma_i} v_1^{\delta_i} \tag{38}$$

with the a_i functions of x distinct from 0.

We put $\lambda_i = \gamma_i + \delta_i$, $\mu_i = \alpha_i + \beta_i$, and, in a plane referred to rectangular axes, plot the points (λ_i, μ_i). We secure thus r or fewer points, each point associated with one or more terms of F.

We consider those of the plotted points which have a least abscissa—say the abscissa ζ_1—and choose from them that point which has a least ordinate—say the ordinate σ_1. For all points (λ_i, μ_i) with $\lambda_i > \zeta_1$, if such exist, we form the ratio

$$\frac{\sigma_1 - \mu_i}{\zeta_1 - \lambda_i}, \tag{39}$$

which is the slope of the straight segment joining (ζ_1, σ_1) to (λ_i, μ_i). Let us suppose that there are segments whose slopes (39) are negative. Taking those segments whose slope is a minimum, we choose the longest of them. Let its right extremity be denoted by (ζ_2, σ_2).

It may be that there are points with $\lambda_i > \zeta_2$ for which

$$\frac{\sigma_2 - \mu_i}{\zeta_2 - \lambda_i} < 0. \tag{40}$$

If so, we take those points which minimize the first member of (40) and choose from them that point (ζ_3, σ_3) whose abscissa is the greatest.

We continue this construction as long as it is possible to secure segments of negative slope. The polygon formed by the segments obtained will be called the *polygon of F*.

If there are no points with $\lambda_i > \zeta_1$ or if there are no such points for which (39) is negative, the polygon of F is defined as the point (ζ_1, σ_1).

When we speak of the points (λ_i, μ_i) lying on a side of a polygon, the extremities of the side will be included.

Consider a point (λ_i, μ_i), plotted for F, which lies on the polygon of F. If there is a term associated with (λ_i, μ_i) which involves either u_1 or v_1, we shall call (λ_i, μ_i) a *b-point*. If no such term exists, that is, if (λ_i, μ_i) is associated with only a single term of F and that term is free of u_1 and v_1, we shall call (λ_i, μ_i) an *a-point*.

Investigation of the Index.

37. We denote the polygon of F by $\mathfrak{P}$. Let (ζ_t, σ_t) be the point of greatest abscissa on $\mathfrak{P}$, that is, the rightmost point on $\mathfrak{P}$. If F is written as in (38), σ_t will be the least of the quantities $\alpha_i + \beta_i$, and ζ_t will be the least value of $\gamma_i + \delta_i$ in those terms of (38) for which $\alpha_i + \beta_i = \sigma_t$.

We are going to work toward the result that *F has an index and the index of F is either ζ_t or ∞.*

38. We begin by showing that if $\mathfrak{P}$ has a b-point, F has strong sums.

We take first the case in which $\mathfrak{P}$ has at least one side. Let l be some side of $\mathfrak{P}$ on which a b-point lies. Let $-\rho$ be the slope of l. We make, in (38), the substitution

$$v = wu^{\rho} \tag{41}$$

with w an indeterminate which admits of differentiation with respect to x. We have

$$v_1 = w_1 u^{\rho} + \rho w u^{\rho-1} u_1. \tag{42}$$

Under (41), a term of (38) associated with a point (λ_i, μ_i) will yield a set of terms, all of degree $\mu_i + \rho\lambda_i$ in u and u_1. If (λ_i, μ_i) is on l, this degree will be the intercept of l on the axis of ordinates. Points not on l produce terms of degree greater than this intercept.

Let A be the sum of those terms of F which correspond to points on l. Under (41), A goes over into a sum

$$\Sigma a_i u^{b_i} u_1^{c_i} \tag{43}$$

with the a_i polynomials in w and w_1. Clearly, if (43) involves u_1 effectively we can fix w as a function ϕ of x so as to make ϕu^{ρ} a strong sum for F.

We shall examine now the case in which (43) is free of u_1. In this case (43) reduces to an expression Bu^q with q the intercept of l and B a polynomial

in w and w_1. Now w_1 must appear effectively in B, for (43) goes over into A by the substitution $w = vu^{-\rho}$.

Let C be an irreducible factor of B which involves w_1 and let

$$B = CD. \tag{44}$$

Let ϕ be a function of x which, substituted for w, annuls C but not $\partial C/\partial w_1$. In (44) we put

$$w = \phi + h, \qquad w_1 = \phi_1 + k$$

with ϕ_1 the derivative of ϕ and h and k indeterminates. The members of (44) become identical polynomials in h and k. The terms of lowest degree produced by C are of the first degree and there is a term βk with $\beta \neq 0$. It follows that B produces a polynomial H in which the terms of lowest degree involve k effectively.

Taking any positive rational number δ, we put

$$h = u^{\delta}, \qquad k = \delta u^{\delta-1}u_1$$

Then H goes over into a finite sum of type (37) in which the terms of lowest degree involve u_1.

All in all, if we make in A, above, the substitution

$$v = \phi u^{\rho} + u^{\rho+\delta}, \tag{45}$$

with ϕ as just fixed and δ rational and positive, A goes over into a sum of the type (37) in which the terms of lowest degree involve u_1.

Suppose now that δ is very small. Then the terms yielded by A under (45) will all have degrees close to the intercept of l and will thus have lower degrees than the other terms produced by F under (45). Thus, the second member of (45) is a strong sum for F.

In the case in which $\mathfrak{P}$ consists of a single point which is a b-point, we start with ρ as any positive rational number and the above argument goes through.

39. From this point until the end of § 47, we assume that $\mathfrak{P}$ has no b-point. Two preliminary questions will be treated in this section.

We shall show that 0 is not a strong sum for F. Let this be false. Then F must have terms free of v and v_1 and, in the sum of such terms, u_1 must figure among the terms of lowest degree. This implies that $\mathfrak{P}$ has a b-point, so that our statement is proved.

Suppose that ζ_1 of § 36 exceeds 0. Then $v = 0$ is an element of F. We

say that $v = 0$ which, according to § 33, has a multiplicity, has the multiplicity ζ_1. It is only necessary to observe, for this, that F has terms free of v_1 and that the minimum of the degrees in v of such terms is ζ_1.

40. Let $\mathfrak{P}$ consist of a single point (ζ_1, σ_1). We say first that F has no strong sum. We know that 0 is not a strong sum. Let the substitution (36) with $\phi_1 \neq 0$ be made in F. The term of F in $u^{\sigma_1} v^{\zeta_1}$ will produce a set of terms of which one, a term in $u^{\sigma_1 + \rho_1 \zeta_1}$, will have a least degree. The familiar intercept argument shows that this term in u alone will have a lower degree than any other term in the sum (37) which F yields.

What just precedes shows also that F can have no element distinct from 0.

The discussion of § 39 and of the present section proves the theorem of § 37 for the case in which $\mathfrak{P}$ consists of a single point, in particular, for the case of $\zeta_t = 0$.

41. Assuming now that $\mathfrak{P}$ has sides, we undertake to determine the possibilities for ρ_1 and ϕ_1 in a strong sum, or in an element distinct from 0.

We shall prove that ρ_1 is the negative of the slope of some side of $\mathfrak{P}$. Let this be false. Let the vertices of $\mathfrak{P}$, arranged according to increasing abscissas, be

(46) $$(\zeta_1, \sigma_1), \cdots, (\zeta_t, \sigma_t).$$

If there are sides of $\mathfrak{P}$ of slopes greater than $-\rho_1$, let (ζ_j, σ_j) be the first point from the left in (46) which is the extremity of such a side. Otherwise let $j = t$. We consider a line through (ζ_j, σ_j) of slope $-\rho_1$. Then all points plotted for F other than (ζ_j, σ_j) lie above this line. It follows, as in § 40, that a substitution $v = \phi_1 u^{\rho_1} + \cdots$ in F produces a non-zero term, free of u_1, which is of lower degree than any other term produced. This proves our statement.

42. Let l be a side of $\mathfrak{P}$ and let $-\rho_1$ be the slope of l. We make in F the substitutions (41), (42), with $\rho = \rho_1$. The terms of F associated with points on l produce, collectively, an expression $L_1(w) u^\tau$ where L_1 is a polynomial in w, and τ the intercept of l on the axis of ordinates. The degree of L_1 is the abscissa of the right extremity of l. The remaining terms of F produce, collectively, an expression M_1 which, arranged as a sum of power products of u and u_1, will have all its terms of degree greater than τ in u and u_1.

Clearly, if $\phi_1 u^{\rho_1}$ is to be the first term of an element or of a strong sum, we must have

(47) $$L_1(\phi_1) = 0.$$

43. Let ϕ_1 be any solution of (47) distinct from zero.[13] We make, in F, the substitution

$$v = \phi_1 u^{\rho_1} + v'. \tag{48}$$

Then F goes over into an expression F' in v' and u. F' will be a polynomial in v' and v'_1, with coefficients which are sums of power products of u and u_1. The exponents of u_1 in the coefficients will be non-negative integers; those of u will be rational numbers and some of them may be negative.

We write F' in the form (38). Now, however, the α_i may be fractional, and even negative. We form a polygon for F' in the manner explained in § 36. We denote this polygon by $\mathfrak{P}'$. We are going to study $\mathfrak{P}'$.

A term in F in

$$u^{\alpha} u_1^{\beta} v^{\gamma} v_1^{\delta} \tag{49}$$

associated with the point $(\gamma + \delta, \alpha + \beta)$, contributes to F' terms coming from

$$u^{\alpha} u_1^{\beta} (v' + \phi_1 u^{\rho_1})^{\gamma} (v'_1 + \phi'_1 u^{\rho_1} + \rho_1 \phi_1 u^{\rho_1 - 1} u_1)^{\delta} \tag{50}$$

with ϕ'_1 the derivative of ϕ_1. Let us consider any term coming from (50). If its degree in v' and v'_1 is $\gamma + \delta - a$ with $0 \leqq a \leqq \gamma + \delta$, its degree in u and u_1 will be $\alpha + \beta + \rho_1 a$. Such a term will be associated with the point

$$(\gamma + \delta - a, \ \alpha + \beta + \rho_1 a). \tag{51}$$

One of the points (51) will be $(\gamma + \delta, \alpha + \beta)$. The others lie on a line sloping upward from that point, with slope $-\rho_1$.

Now, let h represent the point which is the right extremity of l. What precedes makes it geometrically obvious that h and all plotted points on $\mathfrak{P}$ to the right of h are points plotted for F' and, indeed, the lowest plotted points of their respective abscissas. Thus, h is a vertex of $\mathfrak{P}'$ and $\mathfrak{P}$ and $\mathfrak{P}'$ coincide from h onward to the right, that is, they have the same plotted points, which are a-points for both of them.

We shall now examine $\mathfrak{P}'$ to the left of h. Let ϕ_1 be a solution of (47) of multiplicity p. We shall call p the *multiplicity* of ϕ_1. Under the substitution (48), the terms of F associated with points on l will produce collectively the expression [14]

$$L_1(\phi_1 + u^{-\rho_1} v') u^{\tau}. \tag{52}$$

The term of highest degree in v' in (52) will be the term of F' associated with h. The other terms of F' coming from (52) will yield points which lie

[13] The number of such solutions is the length of the horizontal projection of l.

[14] Put $w = \phi_1 + u^{-\rho_1} v'$.

on l or on l produced to the left. The lowest power of v' which figures effectively in (52) is the p-th power. This means that F' has a term in $u^{\tau-\rho_1 p}v'^p$ which is the only term in F' associated with the point $(p, \tau - \rho_1 p)$ and that every other point, plotted for F', of abscissa p, has an ordinate greater than $\tau - \rho_1 p$. Furthermore, all points of abscissa less than p which may be plotted for F' lie in the interior of the upper half-plane determined by l.

It follows that if p is less than the abscissa of h, $\mathfrak{P}'$ has a side of slope $-\rho_1$ which joins $(p, \tau - \rho_1 p)$ to h, and that this side has only a-points.

Whether or not p is less than the abscissa of h, if there are points plotted for F' of abscissa less than p, $\mathfrak{P}'$ has sides of slope less than $-\rho_1$ and $(p, \tau - \rho_1 p)$ is the rightmost extremity of the rightmost such side. Any b-points which $\mathfrak{P}'$ may have lie on sides of slope less than $-\rho_1$.

44. If $\mathfrak{P}'$ has a b-point, lying on a side of slope $-\rho_2 < -\rho_1$, an application of the method of § 38 shows that F has a strong sum of one of the two forms

$$\phi_1 u^{\rho_1} + \phi_2 u^{\rho_2}, \qquad \phi_1 u^{\rho_1} + \phi_2 u^{\rho_2} + u^{\rho_2+\delta}.$$

45. We assume now that $\mathfrak{P}'$ has no b-points. Let q be the abscissa of the leftmost vertex of $\mathfrak{P}'$.

For F' to be annulled by $v' = 0$, that is, for $\phi_1 u^{\rho_1}$ to be an element of F, it is necessary and sufficient that q exceed 0.

Suppose that $q > 0$. Then $\partial^j F'/\partial v'^j$ vanishes for $j < q$ but not for $j = q$. This shows that $\phi_1 u^{\rho_1}$ is an element of F of multiplicity q.

If $q = 0$, we show as in § 39 that $\phi_1 u^{\rho_1}$ is not a strong sum for F.

46. Suppose now that F has an element with at least two terms or a strong sum with at least two terms, the first term, in either case, being $\phi_1 u^{\rho_1}$ above. The arguments of §§ 41, 42 show that $-\rho_2$ is the negative of the slope of a side of $\mathfrak{P}'$ and that ϕ_2 is a root of a certain equation

$$L_2(\phi) = 0 \tag{53}$$

in which the number of solutions distinct from zero, which is the length of the horizontal projection of a side of $\mathfrak{P}'$, does not exceed p of § 43.

47. Let us assume, then, that $\mathfrak{P}'$ has sides of slope less than $-\rho_1$. Let $-\rho_2$ be the slope of some such side. We form the corresponding equation (53) and choose any solution ϕ_2 of (53) which is distinct from 0. We put in F'

$$v' = \phi_2 u^{\rho_2} + v'',$$

whereupon F' goes over into an expression F'' in v''. We form, as above, a

polygon $\mathfrak{P}''$ for F''. If p_1 is the multiplicity[15] of ϕ_2, $\mathfrak{P}''$ will have an a-point—call it h—of abscissa p_1 and, if $\mathfrak{P}''$ has sides of slope less than $-\rho_2$, h will be the rightmost extremity of the rightmost such side. If there are no such sides, h is the leftmost point on $\mathfrak{P}''$. Such b-points as $\mathfrak{P}''$ may have will lie on sides of slope less than $-\rho_2$.

48. We conclude the proof of the result stated in § 37.

If $\mathfrak{P}$ has a b-point, the index of F is ∞.

Suppose that $\mathfrak{P}$ has no b-point. Let r_1 stand for ζ_1 in § 41. By § 39, F will have r_1 (possibly 0) zero elements. There will perhaps be certain possibilities for terms $\phi_1 u^{\rho_1}$ of other elements or of strong sums. The sum of r_1 and of the multiplicities of the ϕ_1 is ζ_t of § 37.

For each $\phi_1 u^{\rho_1}$ we find an F'. If some F' yields a b-point, the index of F is ∞. If no b-points are met, we proceed with each F' as in §§ 45, 47. We find that F has a certain number $r_2 \geqq r$ of zero elements and elements $\phi_1 u^{\rho_1}$ and also, perhaps, a certain number of possibilities $\phi_1 u^{\rho_1} + \phi_2 u^{\rho_2}$ for the beginnings of strong sums or of other elements. The sum of r_2 and of the multiplicities of the ϕ_2 is ζ_t.

At the third step, we form an F'' for each $\phi_2 u^{\rho_2}$. We continue in this manner. There are two ways in which our process, having been carried through k steps, may terminate at the $(k+1)$-th step. Firstly, we may meet an $F^{(k)}$ with a b-point. In that case, the index of F is ∞. Secondly, it may be that no $F^{(k)}$ has a polygon with a side of slope less than the negative of the ρ_k associated with that $F^{(k)}$. In that case, F will have no strong sum and will have precisely ζ_t elements of the types 0 or

$$\phi_1 u^{\rho_1} + \cdots + \phi_h u^{\rho_h}, \qquad h \leqq k,$$

in harmony with § 37.

Let us assume that the process does not terminate in a finite number of steps. Then F has no strong sum and, from some step on, no new finite elements appear. That is, if k is large, we will, in the first k steps, have isolated a fixed number r of finite elements and there may be in addition a finite number of possibilities.

$$\phi_1 u^{\rho_1} + \cdots + \phi_k u^{\rho_k} \tag{54}$$

for the beginnings of elements with an infinite number of terms. The sum of r and of the multiplicities of the ϕ_k, for every large k, is ζ_t. After the finite elements have been isolated, the number of distinct expressions (54) cannot decrease as k increases. Thus, after a certain step, there will be a fixed number

[15] Defined as for ϕ_1.

of distinct expressions (54) and, when an $F^{(k)}$ is determined for any of these expressions, we get a ϕ_{k+1} with the same multiplicity as ϕ_k. An $F^{(k)}$ with k large does not vanish for $v^{(k)} = 0$ and its polygon has just one side of slope less than $-\rho_k$.

This means that we are forming a certain number of infinite series which may be elements of F. Let

$$(55) \qquad \phi_1 u^{\rho_1} + \cdots + \phi_k u^{\rho_k} + \cdots$$

be any one of these infinite series. Let p be the common multiplicity of the ϕ_k in (55) with k large. We shall prove that (55) is an element of F of multiplicity p.

First, we shall show that the ρ_k in (55) have a common denominator. Let, for some large k,

$$(56) \qquad L_k(\phi) = 0$$

be the equation, similar to (53), which determines ϕ_k. Then the degree of L_k in ϕ is p and ϕ_k is a root of (56) of multiplicity p. Thus

$$(57) \qquad L_k(\phi) = a(\phi - \phi_k)^p$$

with a a function of x. This means that the side of $\mathfrak{P}^{(k-1)}$ of slope $-\rho_k$ has on it points, plotted for $F^{(k-1)}$, of each of the abscissas $0, 1, \cdots, p$. If h_0 is that one of these points whose abscissa is 0 and if h_1 is the point of abscissa 1, ρ_k will be the difference of the ordinates of h_0 and h_1. These ordinates are linear combinations of $\rho_1, \cdots, \rho_{k-1}$ and unity, with integral coefficients. Hence ρ_k is such a linear combination. Thus we can use, for the denominator of ρ_k, the common denominator of $\rho_1, \cdots, \rho_{k-1}$.

We prove now that (55) is an element of F of multiplicity p. Let s be the ordinate of the point on $\mathfrak{P}^{(k)}$ of abscissa p. As seen above, s is independent of k for k large.

For any large k, let polygons be formed in the usual manner for the p expressions $\partial^j F^{(k)}/\partial v^{(k)j}$, $j = 0, \cdots, p$. The discussion of (57) shows that the leftmost point of such a polygon will be on the axis of ordinates and will have for ordinate

$$(58) \qquad s + (p - j)\rho_{k+1}.$$

For $j = p$, (58) equals s for every large k, but, for $j < p$, (58) becomes infinite with k. For any j, (58) is the lowest of the degrees in u, u_1 of the terms in the expression obtained by replacing $v^{(k)}$ in $\partial^j F^{(k)}/\partial v^{(k)j}$ by 0. The same expression is obtained on replacing v in $\partial^j F/\partial v^j$ by the sum of the first

k terms in the second member of (55). This shows that (55) is an element of F of multiplicity p.

We have thus established the result stated in § 37.[16]

Multiplicities and Vanishing Derivatives.

49. Suppose that (31) contains certain B_i, say

$$B_1, \cdots, B_d \tag{59}$$

which involve v effectively and which vanish for $u = v = 0$. Let B be any of the B_i in (59).

Referring to Part V, we write

$$S^a F = C_0 B^p + C_1 B^{p_1} B'^{q_1} + \cdots + C_r B^{p_r} B'^{q_r}, \tag{60}$$

with S and B' respectively the separant and derivative of B. The C are forms of order zero in v. Furthermore,

$$p > 0; \; p_i + q_i > p, \qquad (i = 1, \cdots, r).$$

Let a given relation (33) imply $B = 0$. Then the second member of (33) is an element of F. One proves as in *S. S.*, § 55, that (33) gives an element of F of multiplicity p. Furthermore, (33) implies

$$\frac{\partial^{l_1+l_2} F}{\partial v^{l_1} \partial v_1^{l_2}} = 0$$

for $l_2 > 0$, $l_1 + l_2 \leqq p$.

Final Criteria.

50. We develop now a test for determining whether F has either a strong sum, or an element which annuls no B_i in (59) when substituted for v.

Let each B_i in (59) be written as a polynomial in v and let v^{q_i} be the lowest power of v in B_i whose coefficient is not divisible by u. The equation $B_i = 0$ has q_i solutions of the type (33). Let p_i be the value of p in (60) for $B = B_i$. Let

$$m = p_1 q_1 + \cdots + p_d q_d. \tag{61}$$

We compare m with ζ_t.

Suppose that $\zeta_t > m$. Then F has either a strong sum, or an element which annuls no B_i in (59).

Suppose that $\zeta_t < m$. Then ζ_t cannot be the index of F, so that F has a strong sum.

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[16] The matter of the areas in § 35 is handled as in *S. S.*, § 54.

Suppose that $\zeta_t = m$. We shall show, in what follows, how to determine whether the index of F is ζ_t or ∞. If the index is ζ_t, F has no strong sum and every element annuls some B_i.

51. We consider the $q_1 + \cdots + q_d$ solutions of type (33) of the relations $B_i = 0$, $i = 1, \cdots, d$. Let k be a positive integer such that no two of these solutions coincide through their first k terms.

We examine the process of §§ 38-48 for finding elements and strong sums of F.

Let g be any positive integer not greater than k. If the process terminates at the g-th step, it must be either that we have encountered a $\mathfrak{P}^{(g-1)}$ with b-points or that no $\mathfrak{P}^{(g-1)}$ has a side of slope less than the corresponding $-\rho_{g-1}$. The manner of termination would indicate whether the index is ∞ or ζ_t.

Let us suppose that the process does not terminate at the k-th step or at an earlier step. We shall prove that the index is ζ_t.

The non-termination means that we have met certain $\mathfrak{P}^{(k-1)}$ which have no b-points and have sides of slopes less than the associated $-\rho_{k-1}$. There will have been isolated a certain number r_k of elements which are either zero or possess at most $k-1$ terms. The sum of r_k and of the multiplicities of the ϕ_k which the $F^{(k-1)}$ yield is ζ_t.

Thus, $r_k < \zeta_t = m$. Then the B_i must have solutions (33) with at least k terms and the ρ_k of the k-th terms must be negatives of slopes of sides of the $\mathfrak{P}^{(k-1)}$.

Let

$$\phi_1 u^{\rho_1} + \cdots + \phi_k u^{\rho_k} + \cdots, \tag{62}$$

consisting of at least k terms, annul some B_i, say B_1. Let (62) have multiplicity p for F. Then ρ_k in (62) is the negative of the slope of a side of the polygon for some $F^{(k-1)}$ which, for the substitution

$$v^{(k-1)} = \phi_k u^{\rho_k} + v^{(k)},$$

yields an $F^{(k)}$ which is annulled by

$$\phi_{k+1} u^{\rho_{k+1}} + \cdots. \tag{63}$$

One proves now, following *S. S.*, §§ 61, 62, that, if the polygon of $F^{(k)}$ has sides of slope less than $-\rho_k$, p is the abscissa of the right extremity of the rightmost such side. If there are no such sides, p is the least abscissa of the points plotted for $F^{(k)}$. One sees also, as in *S. S.*, § 63, that (63) is zero if and only if p is the least abscissa of the points plotted for $F^{(k)}$.

We suppose that (63) is not zero, so that the polygon $\mathfrak{P}^{(k)}$ of $F^{(k)}$ has at least one side of slope less than $-\rho_k$. We shall prove that $\mathfrak{P}^{(k)}$ has a single such side, namely, a side which has slope $-\rho_{k+1}$ (as in (63)) and has its left end on the axis of ordinates. It will be seen also that this side has no b-points.

Let l be the rightmost side of $\mathfrak{P}^{(k)}$ whose slope is less than $-\rho_k$. As in *S. S.*, § 64, it is seen that the slope of l is not less than $-\rho_{k+1}$.

Suppose now that l has a b-point and let the rightmost such b-point be denoted by h_1. The abscissa of h_1 is less than p.

Suppose first that one of the terms associated with h_1 involves $v_1^{(k)}$; let it be a term in

$$v^{(k)\,l_1}v_1^{(k)\,l_2} \tag{64}$$

with $l_2 > 0$. Let

$$K = \frac{\partial^{l_1+l_2}F^{(k)}}{\partial v^{(k)\,l_1}\partial v_1^{(k)\,l_2}}.$$

By § 49, K is annulled by (63). We form a polygon for K. A point for $F^{(k)}$ yields, for K, a point $l_1 + l_2$ units to the left, or no point, according as the point for $F^{(k)}$ does or does not have a term associated with it which is divisible by (64). As in *S. S.*, § 64, we see that (63) does not annul K.

Suppose now that the terms of $F^{(k)}$ associated with h_1 are all free of $v_1^{(k)}$. Then u_1 appears in those terms. Denoting by q the abscissa of h_1, we let

$$K = \frac{\partial^q F^{(k)}}{\partial v^q}.$$

Then a point plotted for $F^{(k)}$ yields, for K, a point q units to the left, or no point, according as the point for F is or is not associated with a term divisible by v^q. In particular, h_1 yields a point h_2 on the axis of ordinates and the right end of l yields a point which, when joined to h_2, produces a leftmost side, call it l_1, for the polygon of K. The slope of l_1 is that of l and hence is not less than $-\rho_{k+1}$. When (63) is substituted into K, there will result, from the point h_2, terms involving u_1 whose degree in u and u_1 is the ordinate of h_2. The only other terms of K which can conceivably produce terms of this degree are those associated with the points on l_1 other than h_2. Such points are a-points and cannot yield terms which involve u_1. Thus, (63) does not annul K, so that l has no b-point.

The discussion is continued as in *S. S.*, § 64.

An argument like that in *S. S.*, § 65, shows that the index of F is ζ_t. Our discussion has taken care of a set of $F^{(k)}$ for which the sum of the multiplicities of the ϕ_k is at least $\zeta_t - r_k$. This shows that all $F^{(k)}$ are accounted for.

Sufficient Conditions.

52. We denote the general solution of F by $\mathfrak{M}$.

Let F have an element, given by (33). We calculate the successive derivatives of v in (33) formally, expressing each derivative as a polynomial in the derivatives of u with series in u as coefficients.

Suppose that every form which holds $\mathfrak{M}$ vanishes identically in $x, u, u_1, \cdots$, when $v, v_1, \cdots$ are replaced in the form by their expressions found from (33). We shall say, in that case, that the element given by (33) *is in* $\mathfrak{M}$ or *belongs to* $\mathfrak{M}$.

We say that if F has an element which belongs to $\mathfrak{M}$, the solution $u = v = 0$ is in $\mathfrak{M}$. In short, if a form G possesses a term free of the u_i, v_i, the substitution of v as in (33) into G cannot produce zero.

Let us prove that if there are B_i as in (59), no element in $\mathfrak{M}$ can annul any B_i. Let B be any such B_i. Referring to § 31, we denote by R the resultant of B and C_0 with respect to v. Then R, which is a non-zero form in u, vanishes for all solutions of B which are contained in $\mathfrak{M}$. If $G_1, \cdots, G_q$ is a finite set of forms whose manifold is $\mathfrak{M}$, some power of R is a linear combination of $B, G_1, \cdots, G_q$ and their derivatives. Any element in $\mathfrak{M}$ which annuls B would thus annul R. As R does not involve v, R cannot vanish for such an element.

We prove now that an element of F which annuls no B_i in (59) is in $\mathfrak{M}$. Referring to (31), we let

$$T = B_1 B_2 \cdots B_s.$$

If Q is any form which holds $\mathfrak{M}$, QT holds F. It follows that every element of F which does not annul T annuls Q. Now a B_i in (31) which is annulled by an element must involve v and must vanish for $u = v = 0$. Thus an element which annuls no B_i in (59) does not annul T and hence is in $\mathfrak{M}$.

Thus, for $u = v = 0$ to be in $\mathfrak{M}$, it is sufficient for F to have an element which belongs to $\mathfrak{M}$. If F has elements and if there are no B_i as in (59), every element of F is in $\mathfrak{M}$. If there are B_i in (59), an element of F is in $\mathfrak{M}$ if and only if it annuls no B_i.

53. We shall prove now that if F has a strong sum, $u = v = 0$ is in $\mathfrak{M}$. Let F have a strong sum given by (36). We arrange the sum (37) as a polynomial in u_1 and then equate it to zero. We secure an equation

$$A_0 + A_1 u_1 + \cdots + A_q u_1^q = 0 \tag{65}$$

where the A are sums of terms of the form bu^ρ, with b a function of x, and ρ

rational. Regarding (65) as an algebraic equation for u_1, we use the Newton polygon method to form for it solutions of the type

$$(66) \qquad u_1 = \gamma_1 u^{\sigma_1} + \gamma_2 u^{\sigma_2} + \cdots$$

with the γ and σ as usual. Because u_1 figures in the terms of lowest degree in (37), there will be at least one solution (66), (possibly identically zero), with $\sigma_1 \geqq 1$. We consider such a solution (66) and denote by p the common denominator of its σ. Considering (66) as a differential equation, we put $u = w^p$. Then (66) goes over into a differential equation

$$(67) \qquad w_1 = f(x, w)$$

with f analytic for x in some area and w small. Furthermore, $f(x, 0)$ is zero for every x. Equation (67) admits a one-parameter family of solutions

$$w = \psi(x, c)$$

where ψ is analytic for x in some area and for c small, and where ψ, without vanishing identically in x and c, vanishes identically in x for $c = 0$. Representing the p-th power of ψ by ζ, we see that (37) vanishes for $u = \zeta$ if c is small and distinct from 0. For x in a suitable area, we obtain, on replacing u by ζ in (36), a one-parameter family of analytic functions v which tend uniformly towards zero as c approaches zero. We have thus a one-parameter family of solutions of F which approach uniformly, as c decreases, the solution $u = v = 0$. For c small these solutions cannot annul any B in (31). This is because the second member of (36) is not an element of F. Hence, for c small, we get solutions in $\mathfrak{M}$. This proves that $u = v = 0$ is in $\mathfrak{M}$.

Statement of Necessary Conditions.

54. In the sections which follow, we shall prove that if $u = v = 0$ is in $\mathfrak{M}$, at least one of the following conditions is satisfied:

(a) *The manifold of the form u belongs to $\mathfrak{M}$.*
(b) *F has an element which belongs to $\mathfrak{M}$.*
(c) *F has a strong sum.*

That the satisfaction of any one of these conditions insures the presence of $u = v = 0$ in $\mathfrak{M}$ is already known to us.

A NORMALIZATION.

55. It may be necessary later to interchange the letters u and v in F. We represent F by $F(u, v)$. When u and v are interchanged we secure a form which we shall denote by $F(v, u)$.

We are going to prove that if one of the three conditions of § 54 is satisfied by $F(u, v)$, then some one of those conditions is satisfied by $F(v, u)$.

If (a) is satisfied, $F(v, u)$ has the element 0, which belongs to its general solution.

Let (b) be satisfied. If the element described is zero, the general solution of $F(v, u)$ contains the manifold of $u = 0$. If not, the inversion of (33) produces an element of $F(v, u)$ in the general solution of $F(v, u)$. The fact that the coefficients in (33) have a common area of analyticity permits the inversion to be made.

We prove now that if $F(u, v)$ has a strong sum, $F(v, u)$ has a strong sum. Let $F(u, v)$ have a strong sum given by (36). We know from earlier work that if the second member of (36) is zero, then $v = u^\rho$ with ρ sufficiently large and rational is a strong sum for $F(u, v)$. We assume in what follows that the second member of (36) is not zero.

For v as in (36) and for u_1 as in (66), $F(u, v)$ vanishes identically in u and x. Equation (36) defines u as a power series in v,

$$u = \psi_1 v^{\gamma_1} + \psi_2 v^{\gamma_2} + \cdots \tag{68}$$

with $\gamma_1 > 0$. We replace u in (66) by the series in (68). We replace u_1 in (66) by its expression found from (68) by differentiation. This gives a relation between v_1 and v which reduces to a form

$$v_1 = \zeta_1 v^{\delta_1} + \zeta_2 v^{\delta_2} + \cdots \tag{69}$$

with $\delta_1 \geqq 1$. Then $F(u, v)$ vanishes identically in v and x for u as in (68) and v_1 as in (69). On the other hand, because the second member of (36) is not an element, $F(u, v)$ does not vanish identically in x, v, v_1 for u as in (68). When the substitution (68) is made in $F(u, v)$, $F(u, v)$ goes over into a polynomial

$$A_0 + A_1 v_1 + \cdots + A_q v_1^q \tag{70}$$

with the A series in v. Because (70) vanishes when v_1 is taken as in (69), the Newton polygon for (70) must have a side of slope not greater than -1. If, instead of substituting into $F(u, v)$ the entire series in (68), we substitute a sufficiently long segment of the series, we secure an expression similar to

(70) with the same Newton polygon which (70) has; this is because the new expression differs from (70) by terms of high degree in v.

This means that if we take a sufficiently long segment of the series in (68) and replace v in the segment by u, we obtain a strong sum for $F(v, u)$.

56. Continuing with F described as in § 30, we make the assumption that $u = v = 0$ belongs to $\mathfrak{M}$. Suppose that there exists a form, holding $\mathfrak{M}$, of the type

$$u^p + A$$

where A either is identically zero or else has each of its terms of degree higher than p in the u_i, v_i. By Part III, there cannot exist a form holding $\mathfrak{M}$ which is of the type

$$v^q + B$$

where B either is identically zero or else has each of its terms of degree higher than q in the u_i, v_i.

57. Let F be as in § 30 with $u = v = 0$ in $\mathfrak{M}$.

Suppose that F does not involve u. Then $v = 0$ is in the general solution of F considered as a form in v alone. Thus, zero is an element of F in $\mathfrak{M}$ and (b) of § 54 is satisfied. If F does not involve v, (a) is satisfied.

58. On the basis of §§ 55-57, we assume that F, described at the start as in § 30, has $u = v = 0$ in its general solution $\mathfrak{M}$, that F involves both u and v effectively[17] and that no form $u^p + A$ as in § 56 holds $\mathfrak{M}$. It will be proved, in what follows, that one of (b) and (c) of § 54 is satisfied.

Necessity Proof.

59. We denote by S the separant of F for the order v, u of the unknowns.[18] Let p be any positive integer. We consider the forms

(71) $$F, F_1, \cdots, F_p$$

where F_i is the i-th derivative of F.

We now regard the F in (71) as simple forms. The unknowns u_i in the F will be $u_0, \cdots, u_{p'}$, where p' is p or $p + 1$ according as the order of F in u is 0 or 1. The unknowns v_i are $v_0, \cdots, v_{p''}$ with p'' either p or $p + 1$.

[17] The assumption that F involves v is made principally for definiteness of procedure in § 59.

[18] Note that F involves u.

The totality of simple forms which vanish for all solutions of (71) with $S \neq 0$ is a prime system Λ. Λ has the solution $u_i = 0$, $i = 0, \cdots, p'$; $v_i = 0$, $i = 0, \cdots, p''$.

In Λ, we replace each u_i with $i > 0$ by $u_0 z_i$, with z_i a new unknown. We replace v_i, $i = 0, \cdots, p''$ by $u_0 w_i$. Then Λ goes over into a system Ξ in u_0, the z and w. As Λ has solutions with $u_0 \neq 0$, Ξ also has such solutions. The totality Ω of forms in u_0, the z and w which vanish for all solutions of Ξ with $u_0 \neq 0$ is a prime system.

We shall prove that Ω has solutions with $u_0 = 0$. Let this be false. Let $G_1, \cdots, G_q$ be a finite subset of forms of Ω with the same manifold as Ω. Then

$$u_0, G_1, \cdots, G_q$$

has no solutions, so that there exists a relation

$$1 = K_0 u_0 + K_1 G_1 + \cdots + K_q G_q.$$

Then $1 - K_0 u_0$ is a form in Ω. We prove as in *S. S.*, § 69, that Λ has a form M of the type $u_0^g + A$ with each term of A of degree higher than g in the u_i, v_i. M as a differential polynomial holds $\mathfrak{M}$. This contradicts § 58. Thus Ω has solutions with $u_0 = 0$.

60. According to *S. S.*, § 68, there exists a set of functions $\phi(x, h)$, one for each unknown in Ω, analytic for x in some area and for h small, the ϕ for u_0 reducing to 0 for $h = 0$, which annul the forms in Ω but not $u_0 S$.

Passing from Ω to Λ, we have an analogous set of $\phi(x, h)$ for the unknowns in Λ. Let r be the lowest power of h in the expansion of the ϕ which corresponds to u_0. Then the expansion of any other ϕ which is not zero has a least power of h no less than r.

61. Let Φ be some finite system of differential polynomials in u and v, containing F, whose manifold is $\mathfrak{M}$.

We select a value a of x such that:

I. The coefficients in Φ are analytic at a.

II. Every $\phi(x, h)$ for Λ, as in § 60, is analytic for $x = a$ and for h small.

III. The coefficient of h^r in the ϕ corresponding to u_0 does not vanish at a.

IV. S does not vanish identically in h when its unknowns are replaced by their ϕ and x is put equal to a.

62. When x is replaced by a in the ϕ, the ϕ go over into functions of h

which are analytic for h small. We denote the functions of h associated with the u_i by the generic symbol α and those associated with the v_i by β. The α for u_0 has a zero of order r for $h = 0$. Every other α which is not identically zero, and every β which is not identically zero, has a zero of order at least r for $h = 0$. For $x = a$, the F in (71) vanish identically in h when the unknowns are replaced by their α, β.

Let F_i denote for $i > p$, as above for $i \leqq p$, the i-th derivative of F. In F_{p+1}, we replace x by a, the unknowns other than $u_{p'+1}$ and $v_{p''+1}$ by their α and β, and $v_{p''+1}$ by 0. Equating F_{p+1} to zero after these substitutions, we secure a linear equation for $u_{p'+1}$. Because of IV of § 61, $u_{p'+1}$ is determined as a function of h which is either analytic, or else has a pole, for $h = 0$. We treat F_{p+2} similarly, making the substitutions described above and, furthermore, replacing $v_{p''+2}$ by 0 and $u_{p'+1}$ by the function of h found above. We find, for $u_{p'+2}$, a function of h which is either analytic, or has a pole, for $h = 0$.

The process just described determines a one-parameter family of solutions in $\mathfrak{M}$,

$$v = \beta_0(h) + \beta_1(h)(x - a) + \cdots + \frac{\beta_{p''}(h)}{p''!}(x - a)^{p''}, \tag{72}$$

$$u = \alpha_0(h) + \cdots + \frac{\alpha_k(h)}{k!}(x - a)^k + \cdots. \tag{73}$$

The β, and the α_k with $k \leqq p'$, have zeros of orders at least r for $h = 0$. The α_k with $k > p'$ may conceivably have poles for $h = 0$.

63. Let

$$w = \alpha_0(h) + \cdots + \frac{\alpha_{p'}(h)}{p'!}(x - a)^{p'}. \tag{74}$$

Let q be the highest of the orders of the derivatives of u and v which appear in Φ of § 61. We assume that p of § 59 is taken greater than q.

When the second members of (72) and (74) are substituted for v and u respectively in any form G of Φ, G goes over into a series

$$\gamma = \sum_{k=r}^{\infty} \psi_k(x) h^k \tag{75}$$

with the ψ analytic for $x = a$. We prove as in *S. S.*, § 71, that every ψ which is not identically zero has a zero at a of order at least $p - q$.

Representing w in (74) by u, we find, as in *S. S.*, § 72, that u satisfies, for h small, a differential equation

$$u_1 = \mu_0 u + \mu_1 u^{1+1/r} + \cdots + \mu_k u^{1+k/r} + \cdots \tag{76}$$

with μ which are functions of x, analytic for $x = a$. With this same meaning for u, we find for v as given by (72) an expansion

$$v = \nu_0 u + \nu_1 u^{1+1/r} + \cdots + \nu_k u^{1+k/r} + \cdots \tag{77}$$

with ν which are analytic for $x = a$.

64. Following *S. S.*, § 73, we find from (76) and (77), by differentiation, expansions in powers of $u^{1/r}$ for u_i with $i \leqq q$ and v_i with $i \leqq q$ where q is as in § 63. These expansions contain no power of u lower than the first power.

Let these expansions be substituted into any form G in Φ. Then G becomes an expression

$$\zeta_0 u + \zeta_1 u^{1+1/r} + \cdots. \tag{78}$$

As in *S. S.*, § 73, we prove that every ζ which is not identically zero has a zero at a of order at least $p - q$.

As in *S. S.*, § 74, it is possible to find a value a of x which can be used for a sequence of values of p increasing to ∞.

65. Referring to the necessity conditions of § 54, we assume that F, described as in § 58, has no strong sum. We find in §§ 65, 66 an element which belongs to $\mathfrak{M}$.

We take over the discussion of expansions in §§ 76-81 of *S. S.*, speaking here of u-expansions.

Choosing a sequence of values of p which increase towards ∞, we form, for each p, a pair of series of the types shown in (76) and (77). Without loss of generality, we assume that these sequences of u and v, and also the sequence of v_1 which they give by differentiation, have strong characteristics.

When v and u_1 are replaced in F by the corresponding series in (76) and and (77), we secure from F, according to § 64, a sequence of u-expansions converging to zero. We are going to extract from the sequence of v a subsequence which converges to an element of F. Later it will be shown that this element is in $\mathfrak{M}$.

Suppose first that the v converge to zero. We shall prove that zero is an element of F. Let this be false. Then F has terms free of v and v_1. Because F has no strong sum, $\mathfrak{P}$ has no b-points. Thus, among the terms free of v and v_1, there is a term of the type αu^q which is of lower degree than any of the other terms. When v and u_1 are replaced in F by their u-expansions, every term of F which involves v or v_1 produces a sequence of u-expansions con-

verging to zero. The terms free of v and v_1, other than αu^q, produce sequences of characteristics exceeding q. This shows that zero is an element of F.

Now, suppose that the sequence of v has a finite characteristic ρ_1. Of course, $\rho_1 \geqq 1$. We prove, as in *S. S.*, § 84, that $\mathfrak{P}$ has sides and that $-\rho_1$ is the slope of a side. Continuing as in *S. S.*, we find a subsequence of values of p, and a first term $\phi_1 u^{\rho_1}$ of an element of F, such that, for the p of the subsequence, the expansions $v - \phi_1 u^{\rho_1}$ form a sequence of characteristic exceeding ρ_1.

Let F go over into an expression F' in v' and u under the substitution (48). Then the substitution of v and u_1, as in (77) and (76), into F produces the same u-expansions as the substitution of $v' = v - \phi_1 u^{\rho_1}$ and u_1 into F'. Using the fact that the polygon of F' has no b-points, we prove that, if the v' converge to zero, $\phi_1 u^{\rho_1}$ is an element of F. If the sequence of v' has a finite characteristic $\rho_2 > \rho_1$, we continue as in *S. S.*, § 85. Proceeding in this manner, we find a sequence of v as in (77) which converges to an element of F.

66. We shall prove that the element just obtained—call it v_0—belongs to $\mathfrak{M}$.

Let v_0 not belong to $\mathfrak{M}$, and let it annul some form—call it B—of (59). We shall produce the contradiction that F has a strong sum.

We consider the form C_0 of (60) which is secured for B. The solutions of B which belong to $\mathfrak{M}$ annul C_0. Let $\Sigma_1, \cdots, \Sigma_s$ be a decomposition of the system B, C_0 into closed essential irreducible systems. According to § 31, each Σ_i has a basic set

$$U_i, V_i \tag{79}$$

introducing u and v in succession, with U_i algebraically irreducible and of order at most unity in u, and with V_i of order zero in v.

Let us consider those Σ_i of which $u = v = 0$ is a solution.[10] We say that, for at least one of them, U_i is of order unity. Let this be false. If a U_i is of order zero and vanishes for $u = 0$, that U_i, being algebraically irreducible, must be u multiplied by a function of x. Each Σ_i of which $u = v = 0$ is not a solution contains a form $1 + A$ where A vanishes for $u = v = 0$. We conclude that the system B, C_0 is held by a form M given by

$$M = u(1 + H) \tag{80}$$

[10] Because v_0 annuls B, $u = v = 0$ annuls B.

where H vanishes for $u = v = 0$. Thus M holds the system obtained by adjoining B to Φ, so that some M^g is a linear combination of B, the forms of Φ and their derivatives.

We now consider the sequence of u-expansions which converges to v_0. We substitute successively these expansions for v, and the associated expansions for u_1, into the linear expression just found for M^g. Any form of Φ, and any derivative of such a form, will yield a sequence of expansions which converges to zero. Also, for a sequence of expansions converging to v_0, and for the related u_1, B and its derivatives yield sequences which converge to zero. Thus M^g yields a sequence converging to zero. This is impossible because u^g is a term of lowest degree for M^g.[20]

Let then Σ_1 admit $u = v = 0$ as a solution and let U_1 be of order unity. We consider the equation

$$U_1 = 0 \tag{81}$$

as an algebraic equation for u_1. For x in a suitable area $\mathfrak{A}_1$, and for u small, the solutions of (81) can be expressed as series of ascending fractional powers of u with coefficients analytic in $\mathfrak{A}_1$. For x in $\mathfrak{A}_1$ and for u small but not zero, these series give finite values for u_1 which, for given values of x and u, are the only numerical solutions of (81) for u_1. None of these series annuls the initial of V_1 identically in x and u. When any one of the series is substituted for u_1 in V_1, the equation $V_1 = 0$ determines v as any one of a finite number of series of ascending fractional powers of u with coefficients which will be analytic in $\mathfrak{A}_1$ if $\mathfrak{A}_1$ is shrunk appropriately.

We obtain thus a certain number of related pairs of series for u_1 and v. For x in $\mathfrak{A}_1$ and for u small but not zero, such a pair of series gives finite values of u_1 and v, and the pairs of values of u_1 and v from all of the series will be the only numerical solutions of $U_1 = 0$, $V_1 = 0$, for given x and u.

Because u does not hold Σ_1, the solution $u = v = 0$ of Σ_1 is approximable by solutions $\bar{u}$, $\bar{v}$ with $\bar{u} \neq 0$. There are points b in $\mathfrak{A}_1$ for which we can get $\bar{u}$ and $\bar{v}$ which, with any finite number of their derivatives, are as small as one pleases at b, while $\bar{u}(b) \neq 0$. There must be some one related pair of series for u_1 and v which, for some sequence of $\bar{u}$ approximating more and more closely to $u = 0$, give $\bar{u}_1$ and $\bar{v}$ when u is replaced by $\bar{u}$. According to *A. D. E.*, § 89, the lowest exponent of u in the series for u_1 is at least unity. Because $\bar{u}(b)$ and $\bar{v}(b)$ can be made arbitrarily small, the series for $\bar{v}$ must contain only positive exponents of u. Thus there is a pair of series

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[20] It is unnecessary to use strong characteristics here. In *S. S.*, § 81, the characteristic of (159) is not less than $\alpha + \beta$, even if α and β are not strong.

$$u_1 = \alpha_0 u + \alpha_1 u^{1+1/r} + \cdots, \tag{82}$$

$$v = \beta u^{1/r} + \cdots, \tag{83}$$

which annul U_1 and V_1. The remainder of C_0 with respect to the ascending set U_1, V_1 is zero. Because the initial of V_1 does not vanish for (82), C_0 is annulled by (82), (83). Similarly, B is annulled by (83).

We now examine (60).

Let the series in (82) and (83) be denoted by $\bar{u}_1$ and $\bar{v}$. Let C_0 be expanded in powers of $u_1 - \bar{u}_1$ and $v - \bar{v}$, with u-expansions for coefficients. The expression for C_0 will contain no term free of $u_1 - \bar{u}_1$ and $v - \bar{v}$. Because C_0 does not vanish identically in x, u, u_1 for $v = \bar{v}$, the expression contains a term free of $v - \bar{v}$. Let

$$C_0 = \gamma_1 (u_1 - \bar{u}_1)^{g_1} + \cdots + \gamma_t (u_1 - \bar{u}_1)^{g_t} + \cdots \tag{84}$$

where the γ are u-expansions and the g increasing positive integers. The terms which follow $\gamma_t (u_1 - \bar{u}_1)^{g_t}$ in (84) involve $v - \bar{v}$.

In (60), keeping C_0 in its original form as a polynomial in v, u_1, u, we make the substitutions

$$\begin{aligned} v &= \bar{v} + w \\ v_1 &= \frac{\partial \bar{v}}{\partial u} + \frac{\partial \bar{v}}{\partial x} u_1 + w_1 \end{aligned} \tag{85}$$

with w an indeterminate and w_1 its derivative. Then each term in the second member of (60) goes over into a polynomial in w and w_1. The coefficients in these polynomials are polynomials in u_1 whose coefficients are series of rational powers of u. B^p will give a polynomial in w in which the least exponent of w is p. The power products $B^{p_i} B'^{q_i}$ will produce polynomials in w and w_1 with terms all of degree at least $p + 1$.

Let the coefficient of w^p in the polynomial yielded by B^p be a u-expansion in which the lowest exponent of u is h. If we replace w by u^ρ, with ρ a positive integer greater than h, we see that the replacement of v by $\bar{v} + u^\rho$ in B^p produces a u-expansion in which the least exponent of u is $h + p\rho$.

In (84), let the sum

$$\gamma_1 (u_1 - \bar{u}_1)^{g_1} + \cdots + \gamma_t (u - \bar{u}_1)^{g_t} \tag{86}$$

be written as an infinite sum Σ of power products in u_1 and u. Because (86) equals $u_1 - \bar{u}_1$ multiplied by a sum analogous to Σ, and because the terms in $\bar{u}_1$ are all of degree at least unity, the terms of lowest degree in Σ must involve

u_1. Let k be the degree of the terms of lowest degree in Σ. Let ρ above exceed k as well as h. Then the substitution $v = \bar{v} + w^\rho$ into $C_0 B^p$ produces a set of terms in which the terms of lowest degree are of degree $k + h + p\rho$ and involve u_1.

Now the substitution (85) with $w = u^\rho$, practiced on any term after the first in the second member of (60), yields terms in u and u_1 of degree at least $(p+1)\rho$. Let $\rho > h + k$. Then the terms of lowest degree coming from the second member of (60) will involve u_1.

Because S in (60) is free of u_1, v_1, it must be that the substitution (85) with $w = u^\rho$ reduces F to a sum of power products in u and u_1 in which the terms of lowest degree involve u_1. If, then, v' is a sufficiently long segment of $\bar{v}$, the replacement of v in F by $v' + u^\rho$ will produce a sum (37) in which u_1 figures in the terms of lowest degree. Thus $v' + u^\rho$ is a strong sum for F.

Thus v_0 is an element of F in $\mathfrak{M}$. The necessity of the fulfillment of at least one of the conditions of § 54 is established.

Summary of Test.

67. Let us summarize the method for testing whether $u = v = 0$ is contained in $\mathfrak{M}$, the general solution of F. We recall that F is algebraically irreducible, that the maximum of the orders of F in u and v is unity and that $u = v = 0$ is a solution of F.

One first secures the decomposition (31). If there are no B_i as in (31) or no such B_i which are annulled by $u = v = 0$, then $u = v = 0$ is in $\mathfrak{M}$. In what follows, we assume that B_i exist, in (31), which vanish for $u = v = 0$.

One tests now as in Part V to see whether the manifold of the form u is contained in $\mathfrak{M}$. An affimative answer means that $u = v = 0$ is in $\mathfrak{M}$. In what follows, we assume that $\mathfrak{M}$ does not contain the manifold of u.

We compare ζ_t of § 37 with m of § 50, extending the definition of m so as to have $m = 0$ if there are no B_i as in (59). If $\zeta_t \neq m$, $u = v = 0$ is in $\mathfrak{M}$. If $\zeta_t = m$, $u = v = 0$ is or is not in $\mathfrak{M}$ according as the index of F is ∞ or ζ_t; the index is determined as in § 51.[21]

One-Parameter Families.

68. In § 31 it was stated that, if U in (32) is of order unity, the manifold of Σ is contained in $\mathfrak{M}$. We show now how this is proved.

Let $u = \phi$, $v = \psi$ be any solution of Σ.

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[21] If $\zeta_t = 0$, the index is 0. Otherwise the single point of which $\mathfrak{P}$ is composed would be a b-point. This would imply that $\mathfrak{M}$ contains the manifold of u.

Following § 66, we prove that U, V, B, C_0 are each annulled by a pair of series

$$u_1 = \phi_1 + \alpha_0(u - \phi) + \alpha_1(u - \phi)^{1+1/r} + \cdots,$$
$$v = \psi + \beta(u - \phi)^{1/r} + \cdots, \tag{87}$$

with ϕ_1 the derivative of ϕ. We then use (60) to show that the substitution

$$v = \bar{v} + (u - \phi)^{\rho} \tag{88}$$

with $\bar{v}$ the series for v in (87) and ρ a sufficiently large integer, reduces F to a sum of power products in $u - \phi$ and $u_1 - \phi_1$ with $u_1 - \phi_1$ present in the products of lowest degree. The argument of § 53 is then used to show that the solution $u = \phi$, $v = \psi$ of F can be approximated by solutions in $\mathfrak{M}$. Thus $u = \phi$, $v = \psi$ is in $\mathfrak{M}$.

EXAMPLES.

69. *Example* 1. Let

$$A = u(uv_1 + uu_1 - 2u_1v) - (v - u)^2.$$

Let F be the form, algebraically irreducible in the field of all rational functions of x, defined by

$$uF = vA^2 - \prod_{j=0}^{4}\left(v - u + \frac{u^2}{x + j}\right).$$

Then $\mathfrak{P}$ has no b-point, $\zeta_t = 4$, $m = 5$. $\mathfrak{M}$ contains $u = v = 0$.

Example 2. Let

$$A = uv_1 + uu_1 - 2u_1v.$$

Let

$$F = A^2 + \prod_{j=0}^{2}(v - u + ju^2).$$

Here $\mathfrak{P}$ has no b-points and $\zeta_t = m = 3$. Carrying out the substitution $v = u + v'$, we find that $\mathfrak{P}'$ has a b-point, so that $u = v = 0$ is in $\mathfrak{M}$.

Example 3. Let $A = v^2 - u^3$ and let A_1 be the derivative of A. Let

$$F = A_1{}^2 - A.$$

Then $\mathfrak{P}$ has no b-point, and $\zeta_t = m = 2$. By § 51 we see that the index is 2, so that $u = v = 0$ is not in $\mathfrak{M}$.

Example 4. The similarity of the preceding examples to examples of *S. S.* might lead one to ask whether the problem of the present paper cannot

be reduced to that of *S. S.* by replacing v by a form of the first order in u, for instance, by u_1. Let $F = u + v_1^2$. Then $\mathfrak{P}$ has a b-point so that $u = v = 0$ is in $\mathfrak{M}$. If we substitute for v a form of the first order in u which vanishes for $u = 0$, F becomes a form in u for which $u = 0$ is an essential manifold.

Example 5. Let

$$F = v_1^2(v_1 - u_1) + v(v - u).$$

The manifold of F decomposes into $\mathfrak{M}$ and the manifold of v. $\mathfrak{P}$ has no b-point, $\zeta_t = 2$, $m = 1$. We observe that zero and u are elements. By § 41, if F had a strong sum, the first term of the strong sum would be u. If we put, in F,

$$v = u + \phi_2 u^{\rho_2} + \cdots + \phi_k u^{\rho_k}$$

with $\phi_2 \neq 0$, $\rho_2 > 1$, the term $v(v - u)$ in F produces a sum of powers of u in which the least exponent is $\rho_2 + 1$. The first term of F produces power products in u and u_1 of degree no less than $\rho_2 + 2$. There is consequently no strong sum, and the index is 2.

One might ask whether, in the case in which F has a finite index and there are elements in $\mathfrak{M}$, the elements in $\mathfrak{M}$ have to be algebraic with respect to u. At least in the case in which F has constant coefficients, the answer is affirmative. The general question should be interesting to examine.

Example 6. Let $F = v_1^2 + (u_1 + v)v$. The manifold of F consists of $\mathfrak{M}$ and the manifold of v. The two irreducible manifolds have in common the one-parameter family $u = c$, $v = 0$.

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THE ANALYSIS OF THE DIRECT PRODUCT OF IRREDUCIBLE REPRESENTATIONS OF THE SYMMETRIC GROUPS.*

By F. D. Murnaghan.

Let Γ and Γ' denote irreducible representations of the symmetric groups on m and n letters, respectively, with characteristics ϕ and ϕ'. Their direct product $\Gamma \cdot \Gamma'$ is a representation, in general reducible, of the symmetric group on $m+n$ letters with characteristic $\phi\phi'$. We have given in a previous paper **(1)** the analysis of $\Gamma \cdot \Gamma'$ into its irreducible components for all representations Γ, Γ' for which $m+n \leq 9$. In the present paper we present a refinement of the method used in **(1)** which makes the computations very much easier and add the table giving the analyses of the products $\Gamma \cdot \Gamma'$ for which $m+n=10$. The simple characteristics ϕ of the symmetric group have been termed Schur-Functions ($= S$-Functions) and the problem under consideration has been treated recently under the title "Multiplication of S-Functions" by Littlewood and Richardson **(2)** who have proposed a scheme involving the construction of various tableaux (based on the partitions of n with which the irreducible representations of the symmetric group on n letters are associated). They have, however, been unable to present a proof, in the general case, of the theorem on which their proposed scheme rests; and, unfortunately, the example they give to illustrate the operation of their method (namely, the direct product of the irreducible representation $\Gamma = D(4,3,1)$, of dimension 70, of the symmetric group on 8 letters by the irreducible representation $\Gamma' = D(2^2,1)$, of dimension 5, of the symmetric group on 5 letters) has the result incorrectly printed. The tableaux they give furnish the correct result and so the error in the final result must be ascribed to the printer or to an oversight in reading off the representations associated with the various tableaux (of which there are 34). That this error was not immediately detected will not appear surprising when we remark that the direct product being analysed is a representation of dimension 450, 450 of the symmetric group on 13 letters. The method we give in the present paper makes it easy to read off the analysis of this representation in less than five minutes. We furnish rules, with illustrative examples, which make the analysis of $\Gamma \cdot \Gamma'$ simple if $m+n \leq 16$. These rules depend for their construction on the tables, given in our paper **(1)** for $n \leq 9$, which furnish the analysis of the reducible representations $\Delta(\lambda)$ of the symmetric group on n letters; and on tables, reciprocal to these, which express

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each simple characteristic of the symmetric group on n letters as a linear combination of the characters of the reducible representations $\Delta(\lambda)$ of this group. These tables were originally furnished by Kostka ((**3**), (**4**), (**5**)) in connection with his fundamental researches on symmetric functions; his method for constructing them is, however, quite unnecessarily complicated and it is not surprising that the usefulness of his tables for the present problem has hitherto escaped attention. As his tables may not be easily accessible to those working in nuclear physics (for whom, principally, this paper is written) we give the necessary "reciprocal tables" for $n \leq 9$. The calculation of these tables is very simple, involving not more than a few hours work; in fact the simplest construction of the tables of our previous paper, which furnished the analysis of $\Delta(\lambda)$, appears to be the following. First construct the tables of the present paper; each table is a triangular matrix with diagonal elements unity and, hence, having its determinant unity. The tables furnishing the analysis of $\Delta(\lambda)$ are obtained by taking the reciprocals of these triangular matrices; a procedure involving merely a recurrent transposition of terms from one side of a system of linear equations to the other.

Since each simple characteristic ϕ of the symmetric group on n letters is the character of a rational, homogeneous, irreducible representation, of degree n, of the full linear group (= group of all non-singular linear homogeneous transformations in p variables) (**6**) the results of the present paper furnish the analysis of the Kronecker product of any two such rational, homogeneous, irreducible representations of the full linear group.

It is pleasant to conclude this introduction by remarking that our interest in the problem has been greatly stimulated by the queries of our friends Professors J. A. Wheeler and E. Wigner whose fundamental researches in nuclear physics require the analysis of $\Gamma \cdot \Gamma'$ given here.

1. Notations and outline of the general method.[1] Let

$$(\epsilon) = (\epsilon_1, \epsilon_2, \cdots, \epsilon_n), \qquad \epsilon_1 \geq \epsilon_2 \geq \cdots \geq \epsilon_n \geq 0$$

be any partition of n and let $\phi_{(\epsilon)}(s)$ be the simple characteristic of the symmetric group on n letters which is associated with the partition (ϵ); similarly let

$$(\nu) = (\nu_1, \nu_2, \cdots, \nu_m), \qquad \nu_1 \geq \nu_2 \geq \cdots \geq \nu_m \geq 0$$

be a partition of m and let $\phi_{(\nu)}(s)$ be the corresponding simple characteristic of the symmetric group on m letters. Then the product $\phi_{(\epsilon)}(s)\phi_{(\nu)}(s)$ is a

[1] We shall assume the reader familiar with the methods and notations of our previous paper (1) and shall refer, where convenient, to this paper by merely giving the page number.

linear combination of the simple characteristics $\phi_{(\alpha)}(s)$ of the symmetric group on $n+m$ letters. We have given (p. 480) the following rule for determining the coefficients of this linear combination. Regarding $(s)=(s_1, s_2, \cdots)$ as the power sums of $n+m$ variables

$$(z)=(z_1, z_2, \cdots, z_{n+m}) : s_j = z_1^j + \cdots + z_{n+m}^j, \qquad (j=1,2,\cdots)$$

$\phi_{(\epsilon)}(s)$ is a linear combination $\sum\limits_{(\pi)} c_{(\pi)} T_{(\pi)}(z)$ of the symmetric functions $T_{(\pi)}(z)=\Sigma z_1^{\pi_1}\cdots z_n^{\pi_n}$ of degree n in the $n+m$ variables (z) (the summation in the expression for $\phi_{(\epsilon)}(s)$ being over all partitions (π) of n). Then those characteristics $\phi_{(\alpha)}(s)$ of the symmetric group on $n+m$ letters occur for which any member of $[(\alpha)-(\pi)]$ is the partition (ν) of m. It is understood that $[(\alpha)-(\pi)]$ means the aggregate of partitions of m obtained by subtracting (π), in all possible arrangements, from each group of n from the $n+m$ letters (α); and further that any disordered arrangement in the set $[(\alpha)-(\pi)]$ is restored to the normal non-increasing arrangement in the manner described in (**1**) (p. 461). An equivalent statement of this result is clearly the following: add n zeros to (ν) so that it appears as a partition of m with $n+m$ elements $(\nu)=(\nu_1, \nu_2, \cdots, \nu_m, \overbrace{0, 0, \cdots, 0}^{n \text{ zeros}})$ and add (π) in all possible arrangements to each set of n from the $n+m$ numbers (ν). The resulting partitions of $n+m$ are rearranged, if disordered, according to the rule referred to and those which do not vanish will appear in the product $\phi_{(\epsilon)}(s)\phi_{(\nu)}(s)$ with the coefficient $\pm c_{(\pi)}$; the $+$ sign being used if an even, and the $-$ sign if an odd, number of inversions are necessary to bring the disordered partition of $n+m$ into its natural order.

It is clear that a slavish adherence to the rule just given would prove tedious and it would in fact be indicative of a lack of intelligence. For if the expression $\sum\limits_{(\pi)} c_{(\pi)} T_\pi(z)$ be written as a polynomial in z_1 the various coefficients are symmetric functions in the $n+m-1$ variables $z_2, \cdots, z_{n+m}$ the coefficient of z_1^p being of degree $n-p$. Each such symmetric function of degree $n-p$ may be expressed as a linear combination of the simple characteristics of the symmetric group on $n-p$ letters; let $a_{(\lambda')}\phi_{(\lambda')}(s')$ be a term of this linear combination where the s' are the power sums of the $n+m-1$ variables $z_2, \cdots, z_{n+m}$. Then amongst the terms of the desired product $\phi_{(\epsilon)}(s)\phi_{(\nu)}(s)$ will occur terms $a_{(\lambda')}\phi_{(\alpha)}(s)$ where (α) is obtained from any term $\phi_{(\beta)}(s')$ occurring in the product $\phi_{(\nu')}(s')\phi_{(\lambda')}(s')$ (where $(\nu')=(\nu_2, \cdots, \nu_m)$) by prefixing an element ν_1+p. We hope the reader will not be confused by the number of words necessary to state the rule which is really simpler to apply than to describe; a few elementary illustrations will make it clear.

The most elementary (indeed trivial) instance of our problem occurs when $n=1$. Here there is only one partition $(\pi)=(1)$ and only one simple characteristic $\phi_{(\epsilon)}(\boldsymbol{s})=\phi_{(1)}(\boldsymbol{s})=s_1=T_{(1)}(\boldsymbol{z})$ and the original form of statement of our rule is immediately applicable. We write (ν) in the form $(\nu_1,\cdots,\nu_m,0)$ and add, in turn, unity to each element obtaining the result (p. 480)

$$\{1\}\cdot\{\nu_1,\cdots,\nu_m\}=\{\nu_1+1,\cdots,\nu_m\}+\cdots \\ +\{\nu_1,\cdots,\nu_m+1\}+\{\nu_1,\cdots,\nu_m,1\}$$

where we use the notation $\{\nu_1,\cdots,\nu_m\}$ to denote $\phi_{(\nu)}(\boldsymbol{s})$. If $\nu_k>0$, $\nu_{k+1}=\nu_{k+2}=\cdots=\nu_m=0$ we omit the terms on the right in which there appears a unit preceded by a zero (p. 461) and our result appears in the simpler form

$$\{1\}\cdot\{\nu_1,\cdots,\nu_k\}=\{\nu_1+1,\cdots,\nu_k\}+\{\nu_1,\cdots,\nu_k+1\}+\{\nu_1,\cdots,\nu_k,1\}.$$

The next case, namely multiplication by $\{2\}$, is not quite so trivial and enables us to glimpse the advantages of the modification, given above, of the original form of statement. We first remark that since $p_0(\boldsymbol{z})=1$ (p. 449) $\{0,0,0,\cdots\}=1$ so that $\{0,0,0,\cdots\}\cdot\{\nu_1,\cdots,\nu_k\}=\{\nu_1,\cdots,\nu_k\}$. Now $\phi_{(2)}(\boldsymbol{s})=p_2(\boldsymbol{z})=T_{(2)}(\boldsymbol{z})+T_{(1^2)}(\boldsymbol{z})$ and writing this as a polynomial (of the second degree) in z_1 the terms independent of $z_1=\phi_2(\boldsymbol{s}')$; the coefficient of $z_1=\phi_1(\boldsymbol{s}')$ whilst the coefficient of $z_1{}^2$ is unity. Hence

$$\{2\}\cdot\{\nu_1\}=\{2\}\cdot\{\nu_1,0,0,\cdots,0\}=\{\nu_1,2\}+\{\nu_1+1,1\}+\{\nu_1+2\}.$$

From this we deduce

$$\{2\}\cdot\{\nu_1,\nu_2\}=\{\nu_1,\nu_2,2\}+\{\nu_1,\nu_2+1,1\}+\{\nu_1,\nu_2+2\} \\ +\{\nu_1+1,\nu_2,1\}+\{\nu_1+1,\nu_2+1\}+\{\nu_1+2,\nu_2\}$$

and so on in general; the final result being that $\{2\}\cdot\{\nu_1,\nu_2,\cdots,\nu_j\}$ is obtained by writing $\{\nu_1,\nu_2,\cdots,\nu_j\}$ in the form $\{\nu_1,\nu_2,\cdots,\nu_j,0\}$ and then adding 2 and $(1,1)$ in all possible ways (p. 480).

Multiplication by $\{1^2\}$ is even simpler since $\phi_{(1^2)}(\boldsymbol{s})=\sigma_2(\boldsymbol{z})=T_{(1^2)}(\boldsymbol{z})$. Writing this as a polynomial (of the first degree) in z_1 the terms independent of $z_1=\phi_{(1^2)}(\boldsymbol{s}')$ whilst the coefficient of $z_1=\phi_1(s')$. Hence

$$\{1^2\}\cdot\{\nu_1\}=\{\nu_1,1,1\}+\{\nu_1+1,1\}$$

and so on, in general, the final result being that $\{1^2\}\cdot\{\nu_1,\nu_2,\cdots,\nu_j\}$ is obtained by writing $\{\nu_1,\cdots,\nu_j\}$ in the form $\{\nu_1,\cdots,\nu_j,0,0\}$ and then adding $\{1,1\}$ in all possible ways (p. 481).

We hope that it is clear from the preceding paragraphs that the essential steps in the calculation of $\phi_{(\epsilon)}(\boldsymbol{s})\cdot\phi_{(\nu)}(\boldsymbol{s})$ are the following:

(1) First express $\phi_{(\epsilon)}(s)$ in the form $\sum_{(\pi)} c_{(\pi)} T_{(\pi)}(z)$ and write it as a polynomial in z_1. This step in the calculation has already been done for us (as far as $n = 11$) in the tables of Kostka referred to in the introduction; we shall discuss it in the following paragraph.

(2) Next express the coefficients of the various powers of z_1 as linear combinations of the simple characteristics of the appropriate symmetric groups (the coefficient of z_1^p being expressible in terms of the simple characteristics of the symmetric group on $n - p$ letters). This step has also been done for us in Kostka's tables; we give below tables furnishing the necessary coefficients (as far as $n = 9$).

When these steps have been performed the product $\phi_{(\epsilon)}(s) \cdot \phi_{(\nu)}(s)$ can be at once written down if the products $\phi_{(\lambda)}(s) \cdot \phi_{(\nu')}(s)$ are known where $(\nu') = (\nu_2, \cdots, \nu_j)$ and (λ) is a partition of n, or less than n, letters. Thus the desired products are obtainable without difficulty by a recurrence method. Before proceeding to a description of the methods by which steps (1) and (2) are carried out in general it is probably desirable to illustrate the procedure by another simple example.

Example. $\{2, 1\} \cdot \{2, 1\}$.

We shall see below that $\{2, 1\} = T_{(2,1)}(z) + 2T_{(1^3)}(z)$. The terms free of z_1 in this second degree polynomial $= \phi_{(2,1)}(s')$ and so the characteristics beginning with 2 in the desired product are found by prefixing a 2 to the product $\{1\} \cdot \{2, 1\}$ i. e. $\{3, 1\} + \{2^2\} + \{2, 1^2\}$. Since $\{2, 3, 1\} = 0$ (**1**, p. 461) they are, accordingly, $\{2^3\} + \{2^2, 1^2\}$. The coefficient of z_1 in our second degree polynomial is $\Sigma z_2{}^2 + 2\Sigma z_2 z_3 = \phi_2(s') + \phi_{(1^2)}(s')$ and so the characteristics beginning with 3 in the desired product are found by prefixing a 3 to the product $\{1\} \cdot [\{2\} + \{1^2\}]$ i. e. $\{3\} + 2\{2, 1\} + \{1^3\}$. They are, accordingly, $\{3^2\} + 2\{3, 2, 1\} + \{3, 1^3\}$. The coefficient of $z_1{}^2$ is $\phi_1(s')$ and so the characteristics beginning with 4 in the desired product are found by prefixing a 4 to the product $\{1\} \cdot \{1\}$ i. e. $\{2\} + \{1^2\}$. They are, accordingly, $\{4, 2\} + \{4, 1^2\}$. Hence the desired product is

$$\{4, 2\} + \{4, 1^2\} + \{3^2\} + 2\{3, 2, 1\} + \{3, 1^3\} + \{2^3\} + \{2^2, 1^2\}$$

2. Expression of the simple characteristics $\phi_{(\epsilon)}(s)$ of the symmetric group on n letters in terms of the symmetric functions $T_{(\pi)}(z)$. This often treated problem is so simple that it can be discussed *ab initio* in a few lines. $\phi_{(\epsilon)}(s)$ may be presented as a determinant of order n whose diagonal elements are $p_{\epsilon_1}(z), p_{\epsilon_2}(z), \cdots, p_{\epsilon_n}(z)$, the non-diagonal elements being obtained by methodically increasing (decreasing) by unity the labels attached to the $p(z)$ as we move from each column to its neighbor on the right (left). An equiva-

lent statement of this fact is the following: let ξ_j be an operator which decreases by unity the value of the label carrying the subscript j $(j = 1, 2, \cdots, n)$ and let $e_1 = \epsilon_1 + (n-1)$, $e_2 = \epsilon_2 + (n-2), \cdots, e_{n-1} = \epsilon_{n-1} + 1$, $e_n = \epsilon_n$ (so that $e_1 > e_2 > \cdots > e_n \geq 0$). Then

$$\phi_{(\epsilon)}(s) = \begin{vmatrix} \xi_1^{n-1} & \cdots & \xi_n^{n-1} \\ \xi_1^{n-2} & \cdots & \xi_n^{n-2} \\ \cdot & \cdots & \cdot \\ 1 & \cdots & 1 \end{vmatrix} p_{e_1} p_{e_2} \cdots p_{e_n}.$$

If, therefore, we denote by $K_{(\lambda)}(z)$ the characteristic $p_{\lambda_1} \cdots p_{\lambda_n}$ of the reducible representation $\Delta(\lambda)$ of the symmetric group on n letters we see that $\phi_{(\epsilon)}(s)$ is a linear combination of the various compound characteristics $K_{(\lambda)}(z)$; only those characteristics $K_{(\lambda)}(z)$ occurring for which (λ) increased by the set $(n-1, n-2, \cdots, 1, 0)$, arranged in any order, gives the set $(e) = (e_1, \cdots, e_n)$ arranged in any order. The sign attached to $K_{(\lambda)}(z)$ is plus (minus) if the arrangement of the set $(n-1, n-2, \cdots, 1, 0)$ which is added to (λ) to give (e) is even (odd). E. g., let $(\epsilon) = (3, 2, 1)$ so that $(e) = (5, 3, 1)$; to find the $K_{(\lambda)}(z)$ which occur in the expression for $\phi_{(3,2,1)}(s)$ we subtract $(2, 1, 0)$ in all possible orders from $(5, 3, 1)$ obtaining the six terms $(3, 2, 1), -(3, 3, 0)$, $(5, 1, 0)$, $-(5, 2, -1)$, $(4, 3, -1)$, $-(4, 1, 1)$. Of these the fourth and fifth vanish since they contain a negative element (each $p_j(z)$ for which $j < 0$ being zero) and so $\phi_{(3,2,1)} = K_{(3,2,1)} - K_{(3^2)} - K_{(4,1^2)} + K_{(5,1)}$. The same rule holds if we wish to find the expression for any symmetric function $T_{(\lambda)}(z) = \Sigma z_1^{\lambda_1} \cdots z_n^{\lambda_n}$ of degree n in terms of the simple characteristics $\phi_{(\epsilon)}(s)$. In fact $\phi_{(\epsilon)}(s)$ may be presented as the ratio

$$A(e_1, \cdots, e_n) : A(n-1, \cdots, 1, 0)$$

where $A(v_1, \cdots, v_n)$ is the n-rowed determinant of which the elements in the j-th row are the v^j-th powers of the indeterminates $(z_1, \cdots, z_n)$ $(j = 1, \cdots, n)$ (p. 459). On multiplying $T_{(\lambda)}(z)$ by the Vandermonde determinant $\Delta(z) = A(n-1, \cdots, 1, 0)$ we obtain a collection of determinants $A(v_1, \cdots, v_n)$ and on dividing through by $A(n-1, \cdots, 1, 0)$ we see that $T_{(\lambda)}(z)$ is expressible as a linear combination of the simple characteristics $\phi_{(\epsilon)}(s)$; only those simple characteristics $\phi_{(\epsilon)}(s)$ occurring for which (ϵ) increased by the set $(n-1, \cdots, 1, 0)$ i. e. (e) is the same as one of the sets (λ) + any arrangement of the set $(n-1, \cdots, 1, 0)$. The sign attached to $\phi_{(\epsilon)}(s)$ is plus (minus) if the arrangement of the set $(n-1, \cdots, 1, 0)$ which is added to (λ) to give (e) is even (odd). Hence the coefficient of $K_{(\lambda)}(z)$ in the development of $\phi_{(\epsilon)}(s)$ is the same as the coefficient of $\phi_{(\epsilon)}(s)$ in the development of $T_{(\lambda)}(z)$ (a result apparently first observed by Kostka). The second step in our general procedure is therefore easy provided we know

the expansions of the simple characteristics $\phi_{(\epsilon)}(s)$ in terms of the compound characteristics $K_{(\lambda)}(z)$. For partitions (ϵ) containing not more than three non-zero elements these expansions are trivial since they involve nothing more than the expansion of a determinant of order three or less. If (ϵ) has only one non-zero element $\phi_{(\epsilon)}(s)$ appears as a determinant of one row and $\phi_{(\epsilon)}(s) = K_{(\epsilon)}(z)$. E. g., $\phi_{(4)}(s) = K_{(4)}(z)$. For partitions with two non-zero elements we have to expand a two-row determinant; E. g.,

$$\phi_{(4,2)}(s) = \begin{vmatrix} p_4(z) & p_5(z) \\ p_1(z) & p_2(z) \end{vmatrix} = p_4(z)p_2(z) - p_5(z)p_1(z)$$

so that $\phi_{(4,2)}(s) = K_{(4,2)}(z) - K_{(5,1)}(z)$. For three non-zero element partitions we proceed similarly; E. g.,

$$\begin{aligned}\phi_{(3,2^2)}(s) = \begin{vmatrix} p_3(z) & p_4(z) & p_5(z) \\ p_1(z) & p_2(z) & p_3(z) \\ p_0(z) & p_1(z) & p_2(z) \end{vmatrix} &= p_3(z)p_2{}^2(z) - p_3{}^2(z)p_1(z) \\ &\quad - p_4(z)p_2(z)p_1(z) + p_5(z)p_1{}^2(z) \\ &\quad + p_4(z)p_3(z) - p_5(z)p_2(z)\end{aligned}$$

and so

$$\begin{aligned}\phi_{(3,2^2)}(s) &= K_{(3,2^2)}(z) \\ &\quad - K_{(3^2,1)}(z) - K_{(4,2,1)}(z) + K_{(4,3)}(z) + K_{(5,1^2)}(z) - K_{(5,2)}(z).\end{aligned}$$

For partitions containing four or more elements it is convenient to expand $\phi_{(\epsilon)}(s)$ in terms of the first column the cofactors of the elements in this column being simple characteristics corresponding to three element partitions. E. g., suppose we wish $\phi_{(5,2,1^2)}(s)$: from the expression

$$\phi_{(5,2,1^2)}(s) = \begin{vmatrix} p_5 & p_6 & p_7 & p_8 \\ p_1 & p_2 & p_3 & p_4 \\ 0 & p_0 & p_1 & p_2 \\ 0 & 0 & p_0 & p_1 \end{vmatrix}$$

we read $\phi_{(5,2,1^2)}(s) = p_5\phi_{(2,1^2)}(s) - p_1\phi_{(6,1^2)}(s)$ and from the supposed known analyses of $\phi_{(2,1^2)}(s)$ and $\phi_{(6,1^2)}(s)$ we find (since $p_5K_{(2,1^2)}(z) = K_{(5,2,1^2)}(z)$ and so on)

$$\begin{aligned}\phi_{(5,2,1^2)}(s) &= -K_{(8,1)}(z) + K_{(7,1^2)}(z) + K_{(6,2,1)}(z) - K_{(6,1^3)}(z) \\ &\quad + K_{(5,4)}(z) - K_{(5,3,1)}(z) - K_{(5,2^2)}(z) + K_{(5,2,1^2)}(z).\end{aligned}$$

We give below tables furnishing the expressions for the simple characteristics $\phi_{(\epsilon)}(s)$ in terms of the compound characteristics $K_{(\lambda)}(z)$ for all values of n up to 9 inclusive. The simple characteristics are written down the left-hand side of each table (the partitions (ϵ) of n being arranged in dictionary order) and the compound characteristics are written across the top of each table

(the partitions (λ) of n being also arranged in dictionary order). The matrix connecting the $\phi_{(\epsilon)}(s)$ with the $K_{(\lambda)}(z)$ is triangular with diagonal elements all unity and, hence, is of determinant unity; the triangular reciprocal matrix expresses the compound characteristics $K_{(\lambda)}(z)$ in terms of the simple characteristics $\phi_{(\epsilon)}(s)$. These reciprocal matrices have been given, for $2 \leq n \leq 9$, in our paper **(1)** (pp. 475-477). From Kostka's theorem stating that the coefficient of $K_{(\lambda)}(z)$ in the development of $\phi_{(\epsilon)}(s)$ is the same as the coefficient of $\phi_{(\epsilon)}(s)$ in the development of $T_{(\lambda)}(z)$ we see that the coefficients of this latter development are found in the column headed by the partition (λ) in the tables which furnish the expressions for the $\phi_{(\epsilon)}(s)$ in terms of the $K_{(\lambda)}(z)$. As examples of how these tables are read we cite the following:[2]

$$\{1^2\} = -K(2) + K(1^2); \qquad T(2) = \{2\} - \{1^2\}$$
$$\{2,1\} = -K(3) + K(2,1); \qquad T(2,1) = \{2,1\} - 2\{1^3\}$$
$$\{2,1^2\} = K(4) - K(3,1) - K(2^2) + K(2,1^2);$$
$$T(2^2) = \{2^2\} - \{2,1^2\} + \{1^4\}$$
$$\{2^3\} = -K(4,2) + K(4,1^2) + K(3^2) - 2K(3,2,1) + K(2^3);$$
$$T(3,2,1) = \{3,2,1\} - 2\{3,1^3\} - 2\{2^3\} + 4\{2,1^4\} - 6\{1^6\}.$$

If we denote by $c_{(\epsilon)}^{(\lambda)}$ the coefficient of $K_{(\lambda)}(z)$ in the development of $\phi_{(\epsilon)}(s)$:

$$\phi_{(\epsilon)}(s) = \sum_{(\lambda)} c_{(\epsilon)}^{(\lambda)} K_{(\lambda)}(z)$$

Kostka's theorem finds its expression in the formula

$$T_{(\lambda)}(z) = \sum_{(\epsilon)} c_{(\epsilon)}^{(\lambda)} \phi_{(\epsilon)}(s).$$

Expressed in technical terms this says the matrix of the linear transformation from the $\phi_{(\epsilon)}(s)$ to the $T_{(\lambda)}(z)$ is the transpose of the matrix of the linear transformation from the $K_{(\lambda)}(z)$ to the $\phi_{(\epsilon)}(s)$. Since the reciprocal of the transpose of a matrix is the transpose of the reciprocal of the matrix it follows that if we write

$$K_{(\lambda)}(z) = \sum_{(\epsilon)} d_{(\lambda)}^{(\epsilon)} \phi_{(\epsilon)}(s)$$

then

$$\phi_{(\epsilon)}(s) = \sum_{(\lambda)} d_{(\lambda)}^{(\epsilon)} T_{(\lambda)}(z).$$

In other words the coefficients of the expressions for the simple characteristics $\phi_{(\epsilon)}(s)$ in terms of the symmetric functions $T_{(\lambda)}(z)$ are found in the column headed by the partition (λ) in the tables on pp. 475-477 of our paper **(1)**. As examples of how these tables are read we cite the following:

[2] For convenience we denote $\phi_{(\epsilon)}(s)$ by $\{\epsilon\}$; $K_{(\lambda)}(z)$ by $K(\lambda)$; $T_{(\lambda)}(z)$ by $T(\lambda)$.

$$\{2\} = T(2) + T(1^2)\,; \qquad \{2,1\} = T(2,1) + 2T(1^3)$$
$$\{2^2\} = T(2^2) + T(2,1^2) + 2T(1^4)\,;$$
$$\{3,1^2\} = T(3,1^2) + T(2^2,1) + 3T(2,1^3) + 6T(1^5)$$
$$\{3^2\} = T(3^2) + T(3,2,1) + T(3,1^3) + T(2^3) + 2T(2^2,1^2) + 3T(2,1^4) + 5T(1^6)$$

We have, accordingly, in the tables of this and the preceding paper the information necessary to carry out (as far as $n \leq 9$, $m \leq 9$) the steps (1) and (2) of our general method.

3. Tables furnishing the expressions for the simple characteristics $\phi_{(\epsilon)}(s)$ in terms of the compound characteristics $K_{(\lambda)}(z)$ for values of n from 2 to 9 inclusive. In the following tables the $\phi_{(\epsilon)}(s)$ are denoted by $\{\epsilon\}$ and appear down the left-hand side of the table whilst the $K_{(\lambda)}(z)$ are denoted by (λ) and appear across the top of the table. For convenience of printing, Table 8, $n = 9$, is turned around so that the bottom of the page is the left-hand side of the table and the left-hand side of the page the top of the table. The numbers to the right of the main diagonal of each table are all zero and are not written in

1. n = 2.

	(2)	(1^2)
{2}	1	
$\{1^2\}$	−1	1

2. n = 3.

	(3)	(2,1)	(1^3)
{3}	1		
{2,1}	−1	1	
$\{1^3\}$	1	−2	1

3. n = 4.

	(4)	(3,1)	(2^2)	$(2,1^2)$	(1^4)
{4}	1				
{3,1}	−1	1			
$\{2^2\}$	0	−1	1		
$\{2,1^2\}$	1	−1	−1	1	
$\{1^4\}$	−1	2	1	−3	1

4. n = 5.

	(5)	(4,1)	(3,2)	$(3,1^2)$	$(2^2,1)$	$(2,1^3)$	(1^5)
{5}	1						
{4,1}	−1	1					
{3,2}	0	−1	1				
$\{3,1^2\}$	1	−1	−1	1			
$\{2^2,1\}$	0	1	−1	−1	1		
$\{2,1^3\}$	−1	1	2	−1	−2	1	
$\{1^5\}$	1	−2	−2	3	3	−4	1

5. n = 6.

	(6)	(5,1)	(4,2)	$(4,1^2)$	(3^2)	(3,2,1)	$(3,1^3)$	(2^3)	$(2^2,1^2)$	$(2,1^4)$	(1^6)
{6}	1										
{5,1}	−1	1									
{4,2}	0	−1	1								
$\{4,1^2\}$	1	−1	−1	1							
$\{3^2\}$	0	0	−1	0	1						
{3,2,1}	0	1	0	−1	−1	1					
$\{3,1^3\}$	−1	1	1	−1	1	−2	1				
$\{2^3\}$	0	0	−1	1	1	−2	0	1			
$\{2^2,1^2\}$	0	−1	1	1	0	0	−1	−1	1		
$\{2,1^4\}$	1	−1	−2	1	−1	4	−1	1	−3	1	
$\{1^6\}$	−1	2	2	−3	1	−6	4	−1	6	−5	1

6. n = 7.

	(7)	(6,1)	(5,2)	$(5,1^2)$	(4,3)	(4,2,1)	$(4,1^3)$	$(3^2,1)$	$(3,2^2)$	$(3,2,1^2)$	$(3,1^4)$	$(2^3,1)$	$(2^2,1^3)$	$(2,1^5)$	(1^7)
{7}	1														
{6,1}	−1	1													
{5,2}	0	−1	1												
$\{5,1^2\}$	1	−1	−1	1											
{4,3}	0	0	−1	0	1										
{4,2,1}	0	1	0	−1	−1	1									
$\{4,1^3\}$	−1	1	1	−1	1	−2	1								
$\{3^2,1\}$	0	0	1	0	−1	−1	0	1							
$\{3,2^2\}$	0	0	−1	1	1	−1	0	−1	1						
$\{3,2,1^2\}$	0	−1	0	1	1	1	−1	−1	−1	1					
$\{3,1^4\}$	1	−1	−1	1	−2	2	−1	2	1	−3	1				
$\{2^3,1\}$	0	0	1	−1	−1	0	1	2	−1	−2	0	1			
$\{2^2,1^3\}$	0	1	−1	−1	0	0	1	−1	2	1	−1	−2	1		
$\{2,1^5\}$	−1	1	2	−1	2	−4	1	−2	−3	6	−1	3	−4	1	
$\{1^7\}$	1	−2	−2	3	−2	6	−4	3	3	−12	5	−4	10	−6	1

7. n = 8.

	(8)	(7,1)	(6,2)	$(6,1^2)$	(5,3)	(5,2,1)	$(5,1^3)$	(4^2)	(4,3,1)	$(4,2^2)$	$(4,2,1^2)$	$(4,1^4)$	$(3^2,2)$	$(3^2,1^2)$	$(3,2^2,1)$	$(3,2,1^3)$	$(3,1^5)$	(2^4)	$(2^3,1^2)$	$(2^2,1^4)$	$(2,1^6)$	(1^8)
{8}	1																					
{7,1}	−1	1																				
{6,2}	0	−1	1																			
$\{6,1^2\}$	1	−1	−1	1																		
{5,3}	0	0	−1	0	1																	
{5,2,1}	0	1	0	−1	−1	1																
$\{5,1^3\}$	−1	1	1	−1	1	−2	1															
$\{4^2\}$	0	0	0	0	−1	0	0	1														
{4,3,1}	0	0	1	0	0	−1	0	−1	1													
$\{4,2^2\}$	0	0	−1	1	1	−1	0	0	−1	1												
$\{4,2,1^2\}$	0	−1	0	1	0	1	−1	1	−1	−1	1											
$\{4,1^4\}$	1	−1	−1	1	−1	2	−1	−1	2	1	−3	1										
$\{3^2,2\}$	0	0	0	0	−1	1	0	1	−1	−1	0	0	1									
$\{3^2,1^2\}$	0	0	−1	0	1	1	0	0	−1	1	−1	0	−1	1								
$\{3,2^2,1\}$	0	0	1	−1	0	−1	1	−1	3	0	−1	0	−1	−1	1							
$\{3,2,1^3\}$	0	1	0	−1	−1	−1	1	0	0	0	2	−1	2	−1	−2	1						
$\{3,1^5\}$	−1	1	1	−1	2	−2	1	1	−4	−1	3	−1	−2	3	3	−4	1					
$\{2^4\}$	0	0	0	0	−1	2	−1	1	−2	−2	2	0	2	1	−3	0	0	1				
$\{2^3,1^2\}$	0	0	−1	1	1	0	−1	0	−2	2	−1	1	−1	2	1	−2	0	−1	1			
$\{2^2,1^4\}$	0	−1	1	1	0	0	−1	0	2	−2	−1	1	−1	−2	3	2	−1	1	−3	1		
$\{2,1^6\}$	1	−1	−2	1	−2	4	−1	−1	4	3	−6	1	3	−3	−9	8	−1	−1	6	−5	1	
$\{1^8\}$	−1	2	2	−3	2	−6	4	1	−6	−3	12	−5	−3	6	12	−20	6	1	−10	15	−7	1

8. $n = 9$.

	(9)	$(8,1)$	$(7,2)$	$(7,1^2)$	$(6,3)$	$(6,2,1)$	$(6,1^3)$	$(5,4)$	$(5,3,1)$	$(5,2^2)$	$(5,2,1^2)$	$(5,1^4)$	$(4^2,1)$	$(4,3,2)$	$(4,3,1^2)$	$(4,2^2,1)$	$(4,2,1^3)$	$(4,1^5)$	(3^3)	$(3^2,2,1)$	$(3^2,1^3)$	$(3,2^3)$	$(3,2^2,1^2)$	$(3,2,1^4)$	$(3,1^6)$	$(2^4,1)$	$(2^3,1^3)$	$(2^2,1^5)$	$(2,1^7)$	(1^9)
$\{9\}$	1																													
$\{8,1\}$	−1	1																												
$\{7,2\}$	0	−1	1																											
$\{7,1^2\}$	1	−1	−1	1																										
$\{6,3\}$	0	0	−1	0	1																									
$\{6,2,1\}$	0	1	0	−1	−1	1																								
$\{6,1^3\}$	−1	1	1	−1	1	−2	1																							
$\{5,4\}$	0	0	0	0	−1	0	0	1																						
$\{5,3,1\}$	0	0	1	0	0	−1	0	−1	1																					
$\{5,2^2\}$	0	0	−1	1	1	−1	0	0	−1	1																				
$\{5,2,1^2\}$	0	−1	0	1	0	1	−1	1	−1	−1	1																			
$\{5,1^4\}$	1	−1	−1	1	−1	2	−1	−1	2	1	−3	1																		
$\{4^2,1\}$	0	0	0	0	1	0	0	−1	−1	0	0	0	1																	
$\{4,3,2\}$	0	0	0	0	−1	1	0	1	0	−1	0	0	−1	1																
$\{4,3,1^2\}$	0	0	−1	0	0	1	0	1	0	1	−1	0	−1	−1	1															
$\{4,2^2,1\}$	0	0	1	−1	0	−1	1	−1	2	0	−1	0	1	−1	−1	1														
$\{4,2,1^3\}$	0	1	0	−1	0	−1	1	−1	−1	0	2	−1	1	2	−1	−2	1													
$\{4,1^5\}$	−1	1	1	−1	1	−2	1	2	−2	−1	3	−1	−2	−2	3	3	−4	1												
$\{3^3\}$	0	0	0	0	0	0	0	0	−1	1	0	0	1	−2	0	0	0	0	1											
$\{3^2,2,1\}$	0	0	0	0	1	−1	0	−1	0	0	1	0	1	1	−1	−1	0	0	−1	1										
$\{3^2,1^3\}$	0	0	1	0	−1	−1	0	0	1	−1	1	0	0	0	−1	2	−1	0	1	−2	1									
$\{3,2^3\}$	0	0	0	0	−1	2	−1	1	−1	−1	1	0	−1	0	2	−1	0	0	1	−2	0	1								
$\{3,2^2,1^2\}$	0	0	−1	1	0	1	−1	1	−1	1	−2	1	−2	0	3	1	−1	0	0	0	−1	−1	1							
$\{3,2,1^4\}$	0	−1	0	1	1	1	−1	0	0	0	−2	1	1	−2	−1	−1	3	−1	−1	4	−1	1	−3	1						
$\{3,1^6\}$	1	−1	−1	1	−2	2	−1	−2	4	1	−3	1	2	4	−6	−3	4	−1	1	−6	4	−1	6	−5	1					
$\{2^4,1\}$	0	0	0	0	1	−2	1	−1	0	1	1	−1	2	0	−4	−1	2	0	−1	4	1	−1	−3	0	0	1				
$\{2^3,1^3\}$	0	0	1	−1	−1	0	1	0	2	−2	1	−1	0	0	−1	2	−2	1	1	−5	2	2	3	−2	0	−2	1			
$\{2^2,1^5\}$	0	1	−1	−1	0	0	1	0	−2	2	1	−1	−1	2	4	−3	−2	1	0	0	−3	−3	3	3	−1	3	−4	1		
$\{2,1^7\}$	−1	1	2	−1	2	−4	1	2	−4	−3	6	−1	−2	−6	6	9	−8	1	−1	9	−4	4	−18	10	−1	−4	10	−6	1	
$\{1^9\}$	1	−2	−2	3	−2	6	−4	−2	6	3	−12	5	3	6	−12	−12	20	−6	1	−12	10	−4	30	−30	7	5	−20	21	−8	1

4. Rules for the analysis of the direct product with illustrative examples. If (ϵ') is the associate partition of n to (ϵ) $\phi_{(\epsilon')}(s)$ is obtained from $\phi_{(\epsilon)}(s)$ by changing the signs of $s_2, s_4, \cdots$ (p. 453). Hence the analysis of $\phi_{(\epsilon')}(s)\phi_{(\nu')}(s)$ follows from that of $\phi_{(\epsilon)}(s)\phi_{(\nu)}(s)$ by taking the associates of all the partitions found in the latter. Thus from

$$\{2, 1\} \cdot \{1^2\} = \{3, 2\} + \{3, 1^2\} + \{2^2, 1\} + \{2, 1^3\}$$

we read

$$\{2, 1\} \cdot \{2\} = \{4, 1\} + \{3, 2\} + \{3, 1^2\} + \{2^2, 1\}.$$

In calculating $\{\epsilon\} \cdot \{\nu\}$ it is convenient to take $n \leq m$ and to have (ϵ) preceded by (if not identical with) its associate (ϵ'); otherwise we calculate first $\{\epsilon'\} \cdot \{\nu'\}$ and read off from this the desired result. The obvious reason for this is that the tables of the preceding paragraph and of **(1)** pp. 475-477 are simpler as we proceed farther out in our ordered set of partitions. As we have seen in § 2 the general procedure in analysing $\{\epsilon\} \cdot \{\nu\}$ may be formalised as follows

A. Precede by ν_1 each partition of the, supposed known, analysis of $\{\nu_2, \cdots, \nu_m\}$ by $\{\epsilon\}$. This step is the same for all $\{\epsilon\}$, $\{\nu\}$ and we need say nothing further about it.

B. Precede by $\nu_1 + 1$ each partition of the, supposed known, analysis of the product of $\{\nu_2, \cdots, \nu_m\}$ by a linear combination of simple characteristics of the symmetric group on $n - 1$ letters. This linear combination is determined, by the method explained in § **2**, from the tables of § **3** and of **(1)** pp. 475-477. We shall denote it by the symbol B and shall give below tables furnishing B for all partitions (ϵ) of $n \leqq 8$ for which (ϵ) follows, if it is not identical with, its associate (ϵ').

C. Precede by $\nu_1 + 2$ each partition of the, supposed known, analysis of the product of $\{\nu_2, \cdots, \nu_m\}$ by a linear combination of the symmetric group on $n - 2$ letters. We denote this linear combination by C and give it for the same partitions (ϵ) as before.

D. Same as B save that $\nu_1 + 1$ and $n - 1$ are replaced by $\nu_1 + 3$ and $n - 3$, respectively.

E. Same as B save that $\nu_1 + 1$ and $n - 1$ are replaced by $\nu_1 + 4$ and $n - 4$, respectively,

and so on.

Tables furnishing the linear combinations *B, C, D, E*

1. n = 2.

	B
$\{1^2\}$	$\{1\}$

2. n = 3.

	B	C
$\{1^3\}$	$\{1^2\}$	
$\{2,1\}$	$\{2\} + \{1^2\}$	$\{1\}$

3. n = 4.

	B	C
$\{1^4\}$	$\{1^3\}$	
$\{2,1^2\}$	$\{2,1\} + \{1^3\}$	$\{1^2\}$
$\{2^2\}$	$\{2,1\}$	$\{2\}$

4. n = 5.

	B	C	D
$\{1^5\}$	$\{1^4\}$		
$\{2,1^3\}$	$\{2,1^2\} + \{1^4\}$	$\{1^3\}$	
$\{2^2,1\}$	$\{2^2\} + \{2,1^2\}$	$\{2,1\}$	
$\{3,1^2\}$	$\{3,1\} + \{2,1^2\}$	$\{2,1\} + \{1^3\}$	$\{1^2\}$

5. n = 6.

	B	C	D
$\{1^6\}$	$\{1^5\}$		
$\{2,1^4\}$	$\{2,1^3\} + \{1^5\}$	$\{1^4\}$	
$\{2^2,1^2\}$	$\{2^2,1\} + \{2,1^3\}$	$\{2,1^2\}$	
$\{2^3\}$	$\{2^2,1\}$	$\{2^2\}$	
$\{3,1^3\}$	$\{3,1^2\} + \{2,1^3\}$	$\{2,1^2\} + \{1^4\}$	$\{1^3\}$
$\{3,2,1\}$	$\{3,2\} + \{3,1^2\} + \{2^2,1\}$	$\{3,1\} + \{2^2\} + \{2,1^2\}$	$\{2,1\}$

6. n = 7.

	B	C	D	E
$\{1^7\}$	$\{1^6\}$			
$\{2,1^5\}$	$\{2,1^4\} + \{1^6\}$	$\{1^5\}$		
$\{2^2,1^3\}$	$\{2^2,1^2\} + \{2,1^4\}$	$\{2,1^3\}$		
$\{2^3,1\}$	$\{2^3\} + \{2^2,1^2\}$	$\{2^2,1\}$		
$\{3,1^4\}$	$\{3,1^3\} + \{2,1^4\}$	$\{2,1^3\} + \{1^5\}$	$\{1^4\}$	
$\{3,2,1^2\}$	$\{3,2,1\} + \{3,1^3\} + \{2^2,1^2\}$	$\{3,1^2\} + \{2^2,1\} + \{2,1^3\}$	$\{2,1^2\}$	
$\{3,2^2\}$	$\{3,2,1\} + \{2^3\}$	$\{3,2\} + \{2^2,1\}$	$\{2^2\}$	
$\{4,1^3\}$	$\{4,1^2\} + \{3,1^3\}$	$\{3,1^2\} + \{2,1^3\}$	$\{2,1^2\} + \{1^4\}$	$\{1^3\}$

7. n = 8.

	B	C	D	E
$\{1^8\}$	$\{1^7\}$			
$\{2,1^6\}$	$\{2,1^5\} + \{1^7\}$	$\{1^6\}$		
$\{2^2,1^4\}$	$\{2^2,1^3\} + \{2,1^5\}$	$\{2,1^4\}$		
$\{2^3,1^2\}$	$\{2^3,1\} + \{2^2,1^3\}$	$\{2^2,1^2\}$		
$\{2^4\}$	$\{2^3,1\}$	$\{2^3\}$		
$\{3,1^5\}$	$\{3,1^4\} + \{2,1^5\}$	$\{2,1^4\} + \{1^6\}$	$\{1^5\}$	
$\{3,2,1^3\}$	$\{3,2,1^2\} + \{3,1^4\} + \{2^2,1^3\}$	$\{3,1^3\} + \{2^2,1^2\} + \{2,1^4\}$	$\{2,1^3\}$	
$\{3,2^2,1\}$	$\{3,2^2\} + \{3,2,1^2\} + \{2^3,1\}$	$\{3,2,1\} + \{2^3\} + \{2^2,1^2\}$	$\{2^2,1\}$	
$\{3^2,1^2\}$	$\{3^2,1\} + \{3,2,1\}$	$\{3,2,1\} + \{3,1^3\}$	$\{3,1^2\}$	
$\{3^2,2\}$	$\{3^2,1\} + \{3,2^2\}$	$\{3^2\} + \{3,2,1\}$	$\{3,2\}$	
$\{4,1^4\}$	$\{4,1^3\} + \{3,1^4\}$	$\{3,1^3\} + \{2,1^4\}$	$\{2,1^3\} + \{1^5\}$	$\{1^4\}$
$\{4,2,1^2\}$	$\{4,2,1\} + \{4,1^3\} + \{3,2,1^2\}$	$\{4,1^2\} + \{3,2,1\} + \{3,1^3\} + \{2^2,1^2\}$	$\{3,1^2\} + \{2^2,1\} + \{2,1^3\}$	{2,1[illegible]

Illustrative examples. Since we give below the complete table analysing the direct product $\Gamma \cdot \Gamma'$ for the cases $m + n = 10$ we shall illustrate by examples for which $m + n > 10$. We arrange the notation so that $n \leq m$ and if (ϵ) precedes its associate partition (ϵ') we first multiply the associated representations and then take the associates of all terms occurring in the product.

Example 1. $\{2, 1\} \cdot \{4, 2^2\}$.

We read from Table 6 (**1**, p. 484)

$$\{2^2\} \cdot \{2, 1\} = \{4, 3\} + \{4, 2, 1\} + \{3^2, 1\} + \{3, 2^2\} + \{3, 2, 1^2\} + \{2^3, 1\}$$

and prefix a 4 to each of the partitions of 7 which occur on the right. From Table 2 above we read $B = \{2\} + \{1^2\}$ and from Table 5 (**1**, p. 484) we read

$$\{2^2\} \cdot [\{2\} + \{1^2\}] = \{4, 2\} + \{3^2\} + 2\{3, 2, 1\} + \{2^3\} + \{2^2, 1^2\};$$

we then prefix a 5 to each of the partitions of 6 which occur on the right. Again from Table 2 above we read $C = \{1\}$ and from Table 4 (**1**, p. 483) $\{2^2\}\{1\} = \{3, 2\} + \{2^2, 1\}$; we prefix a 6 to each of the partitions of 5 which occurs on the right. Hence the desired analysis of the direct product $\Gamma_{(2,1)} \cdot \Gamma_{(4,2^2)}$ is

$$\begin{aligned} \{2, 1\} \cdot \{4, 2^2\} = {} & \{6, 3, 2\} + \{6, 2^2, 1\} + \{5, 4, 2\} + \{5, 3^2\} + 2\{5, 3, 2, 1\} \\ & + \{5, 2^3\} + \{5, 2^2, 1^2\} + \{4^2, 3\} + \{4^2, 2, 1\} + \{4, 3^2, 1\} \\ & + \{4, 3, 2^2\} + \{4, 3, 2, 1^2\} + \{4, 2^3, 1\}. \end{aligned}$$

As a check against possible errors in copying from the tables the dimensions of the representations on both sides should be calculated. Thus $\{2, 1\}$ is the characteristic of an irreducible representation, of dimension 2, of the symmetric group on 3 letters whilst $\{4, 2^2\}$ is the characteristic of an irreducible representation, of dimension 56, of the symmetric group on 8 letters. Hence, (**1**, p. 448) $\{2, 1\} \cdot \{4, 2^2\}$ is the characteristic of a reducible representation, of dimension $2 \cdot 56 \cdot 11! \div 3!\,8! = 2^4 \cdot 3 \cdot 5 \cdot 7 \cdot 11 = 18{,}480$, of the symmetric group on 11 letters. The dimensions of the various irreducible representations of the symmetric group on 11 letters which occur in the analysis of $\{2, 1\} \cdot \{4, 2^2\}$ given above may be calculated by means of the formula of Frobenius (**1** (11) p. 460) or read off from the table (**7**, pp. 201-204). On removing, for convenience, the common factor 11 we obtain the check

$$1{,}680 = 90 + 100 + 90 + 60 + 420 + 75 \\ + 140 + 42 + 120 + 108 + 120 + 210 + 105.$$

Example 2. $\{3,1\}\cdot\{7,2\}$.

Here $(3,1)$ precedes its associate $(2,1^2)$ and so we calculate $\{2,1^2\}\cdot\{2^2,1^5\}$. Reading the tables as explained in detail in the previous example we have first to calculate $\{2,1^2\}\cdot\{2,1^5\}$. To do this we read off

$$\{2,1^2\}\{1^5\} = \{3,2^2,1^2\} + \{3,2,1^4\} + \{3,1^6\} + \{2^3,1^3\} + \{2^2,1^5\} + \{2,1^7\}.$$

On prefixing a 2 the terms arising from those partitions of 9 on the right which begin with a 3 vanish since $\{2,3,2^2,1^2\} = 0$ (**1**, p. 461) etc. Hence the terms in the analysis of $\{2,1^2\}\cdot\{2,1^5\}$ which begin with a 2 are $\{2^4,1^3\} + \{2^3,1^5\} + \{2^2,1^7\}$. To find the terms beginning with a 3 we read off

$$[\{2,1\} + \{1^3\}]\{1^5\} = \{3,2,1^3\} + \{3,1^5\} \\ + \{2^3,1^2\} + 2\{2^2,1^4\} + 2\{2,1^6\} + \{1^8\}$$

so that the terms beginning with a 3 are

$$\{3^2,2,1^3\} + \{3^2,1^5\} + \{3,2^3,1^2\} + 2\{3,2^2,1^4\} + 2\{3,2,1^6\} + \{3,1^8\}.$$

The terms beginning with a 4 are found from

$$\{1^2\}\cdot\{1^5\} = \{2^2,1^3\} + \{2,1^5\} + \{1^7\}$$

and are

$$\{4,2^2,1^3\} + \{4,2,1^5\} + \{4,1^7\}.$$

Hence we have

$$\{2,1^2\}\cdot\{2,1^5\} = \{4,2^2,1^3\} + \{4,2,1^5\} + \{4,1^7\} + \{3^2,2,1^3\} + \{3^2,1^5\} \\ + \{3,2^3,1^2\} + 2\{3,2^2,1^4\} + 2\{3,2,1^6\} + \{3,1^8\} \\ + \{2^4,1^3\} + \{2^3,1^5\} + \{2^2,1^7\}$$

(a preliminary calculation which would have been unnecessary if we had already prepared the table for $n + m = 11$). The terms beginning with a 2 in the analysis of $\{2,1^2\}\cdot\{2^2,1^5\}$ are accordingly

$$\{2,4,2^2,1^3\} + \{2,4,2,1^5\} + \{2,4,1^7\} + \{2^5,1^3\} + \{2^4,1^5\} + \{2^3,1^7\} \\ = -\{3^2,2^2,1^3\} - \{3^2,2,1^5\} - \{3^2,1^7\} + \{2^5,1^3\} + \{2^4,1^5\} + \{2^3,1^7\}$$

(the terms beginning with 3, in the analysis of $\{2, 1^2\} \cdot \{2, 1^5\}$ vanishing when the 2 is prefixed). To find the terms beginning with 3 we read off, from the table below, $[\{2, 1\} + \{1^3\}] \cdot \{2, 1^5\}$ obtaining, on prefixing a 3,

$$\{3^3, 1^4\} + 2\{3^2, 2^2, 1^3\} + 3\{3^2, 2, 1^5\} + 2\{3^2, 1^7\} \\ + \{3, 2^4, 1^2\} + 2\{3, 2^3, 1^4\} + 2\{3, 2^2, 1^6\} + \{3, 2, 1^8\}.$$

To find the terms beginning with 4 we read off $\{1^2\} \cdot \{2, 1^5\}$ and obtain, on prefixing a 4,

$$\{4, 3, 2, 1^4\} + \{4, 2^3, 1^3\} + \{4, 3, 1^6\} + \{4, 2^2, 1^5\} + \{4, 2, 1^7\}.$$

Collecting our results we find

$$\begin{aligned}\{2, 1^2\} \cdot \{2^2, 1^5\} = {} & \{4, 3, 2, 1^4\} + \{4, 2^3, 1^3\} + \{4, 3, 1^6\} + \{4, 2^2, 1^5\} \\ & + \{4, 2, 1^7\} + \{3^3, 1^4\} + \{3^2, 2^2, 1^3\} + 2\{3^2, 2, 1^5\} \\ & + \{3^2, 1^7\} + \{3, 2^4, 1^2\} + 2\{3, 2^3, 1^4\} + 2\{3, 2^2, 1^6\} \\ & + \{3, 2, 1^8\} + \{2^5, 1^3\} + \{2^4, 1^5\} + \{2^3, 1^7\}.\end{aligned}$$

On taking the associates of the representations occurring in this analysis we obtain

$$\begin{aligned}\{3, 1\} \cdot \{7, 2\} = {} & \{10, 3\} + \{10, 2, 1\} + \{9, 4\} + 2\{9, 3, 1\} + \{9, 2^2\} \\ & + \{9, 2, 1^2\} + \{8, 5\} + 2\{8, 4, 1\} + 2\{8, 3, 2\} + \{8, 3, 1^2\} \\ & + \{8, 2^2, 1\} + \{7, 5, 1\} + \{7, 4, 2\} + \{7, 4, 1^2\} \\ & + \{7, 3^2\} + \{7, 3, 2, 1\}.\end{aligned}$$

On dividing out by the common factor 13 we have the check by dimensions:

$$\begin{aligned}4455 = {} & 16 + 33 + 33 + 210 + 72 + 110 + 44 + 396 \\ & + 528 + 324 + 280 + 220 + 462 + 528 + 275 + 924.\end{aligned}$$

Example 3. $\{2^2, 1\} \cdot \{4, 3, 1\}$.

This is the example, referred to in the introduction, of which the analysis was wrongly printed in (**2**). We read off

$$\begin{aligned}\{2^2, 1\} \cdot \{3, 1\} = {} & \{5, 3, 1\} + \{5, 2^2\} + \{5, 2, 1^2\} + \{4, 3, 2\} + \{4, 3, 1^2\} \\ & + 2\{4, 2^2, 1\} + \{4, 2, 1^3\} + \{3^2, 2, 1\} + \{3, 2^3\} + \{3, 2^2, 1^2\}\end{aligned}$$

so that the terms in the desired analysis beginning with 4 are

$$\{4^2, 3, 2\} + \{4^2, 3, 1^2\} + 2\{4^2, 2^2, 1\} + \{4^2, 2, 1^3\} \\ + \{4, 3^2, 2, 1\} + \{4, 3, 2^3\} + \{4, 3, 2^2, 1^2\}.$$

Also we read off

$$[\{2^2\} + \{2, 1^2\}] \cdot \{3, 1\} = \{5, 3\} + 2\{5, 2, 1\} + \{5, 1^3\} + 2\{4, 3, 1\} + 2\{4, 2^2\} + 3\{4, 2, 1^2\} + \{4, 1^4\} + \{3^2, 2\} + \{3^2, 1^2\} + 2\{3, 2^2, 1\} + \{3, 2, 1^3\}$$

and the terms in the desired analysis beginning with 5 are obtained by prefixing a 5 to the partitions of 8 which appear on the right-hand side. Finally

$$\{2, 1\} \cdot \{3, 1\} = \{5, 2\} + \{5, 1^2\} + \{4, 3\} + 2\{4, 2, 1\} + \{4, 1^3\} + \{3^2, 1\} + \{3, 2^2\} + \{3, 2, 1^2\}$$

and the terms in the desired analysis beginning with 6 are found by prefixing a 6 to the partitions of 7 which appear on the right-hand side. Collecting terms we obtain

$$\{2^2, 1\} \cdot \{4, 3, 1\} = \{6, 5, 2\} + \{6, 5, 1^2\} + \{6, 4, 3\} + 2\{6, 4, 2, 1\} + \{6, 4, 1^3\} + \{6, 3^2, 1\} + \{6, 3, 2^2\} + \{6, 3, 2, 1^2\} + \{5^2, 3\} + 2\{5^2, 2, 1\} + \{5^2, 1^3\} + 2\{5, 4, 3, 1\} + 2\{5, 4, 2^2\} + 3\{5, 4, 2, 1^2\} + \{5, 4, 1^4\} + \{5, 3^2, 2\} + \{5, 3^2, 1^2\} + 2\{5, 3, 2^2, 1\} + \{5, 3, 2, 1^3\} + \{4^2, 3, 2\} + \{4^2, 3, 1^2\} + 2\{4^2, 2^2, 1\} + \{4^2, 2, 1^3\} + \{4, 3^2, 2, 1\} + \{4, 3, 2^3\} + \{4, 3, 2^2, 1^2\}.$$

On dividing out by the common factor 13×11 we obtain the check by dimensions

$$3150 = 36 + 40 + 45 + 240 + 72 + 80 + 84 + 144 + 24 + 120 + 35 + 210 + 180 + 450 + 63 + 81 + 112 + 300 + 144 + 60 + 81 + 180 + 84 + 105 + 60 + 120.$$

Example 4. $\{2^4\} \cdot \{4^2\}$.

For the first set of terms we have to evaluate $\{2^4\} \cdot \{4\}$ and for this we calculate its associate:

$$\{1^4\} \cdot \{4^2\} = \{5^2, 1^2\} + \{5, 4, 1^3\} + \{4^2, 1^4\}$$

so that

$$\{2^4\} \cdot \{4\} = \{6, 2^3\} + \{5, 2^3, 1\} + \{4, 2^4\}.$$

The first set of terms follows by prefixing a 4 and is — $\{5^2, 2^3\} + \{4^2, 2^4\}$. The second set of terms requires the evaluation of $\{2^3, 1\} \cdot \{4\}$; the associate of this,

$$\{1^4\} \cdot \{4, 3\} = \{5, 4, 1^2\} + \{5, 3, 1^3\} + \{4^2, 1^3\} + \{4, 3, 1^4\}$$

so that

$$\{2^3, 1\} \cdot \{4\} = \{6, 2^2, 1\} + \{5, 2^3\} + \{5, 2^2, 1^2\} + \{4, 2^3, 1\}.$$

Hence the second group of terms is

$$\{5^2, 2^3\} + \{5^2, 2^2, 1^2\} + \{5, 4, 2^3, 1\}.$$

The third and last set of terms requires the product

$$\{2^3\} \cdot \{4\} = \{6, 2^2\} + \{5, 2^2, 1\} + \{4, 2^3\}$$

and is

$$\{6^2, 2^2\} + \{6, 5, 2^2, 1\} + \{6, 4, 2^3\}.$$

Collecting we have

$$\begin{aligned}\{2^4\} \cdot \{4^2\} = \{6^2, 2^2\} &+ \{6, 5, 2^2, 1\} + \{6, 4, 2^3\} \\ &+ \{5^2, 2^2, 1^2\} + \{5, 4, 2^3, 1\} + \{4^2, 2^4\}.\end{aligned}$$

A partial check on the accuracy of the analysis is furnished by the fact that the six representations occurring on the right consist of three pairs of associated representations so that the direct product $\Gamma_{(2^4)} \cdot \Gamma_{(4^2)}$ is self-associated, as it must be. On removing the common factor $13 \cdot 11 \cdot 7 \cdot 5 \cdot 3 \cdot 2$ we obtain the check by dimensions $84 = 5 + 21 + 16 + 16 + 21 + 5$. The direct product being analysed in this example is a representation, of dimension $13 \cdot 11 \cdot 7^2 \cdot 5 \cdot 3^2 \cdot 2^3 = 2{,}522{,}520$ of the symmetric group on 16 letters.

5. Table furnishing the analysis of $\Gamma . \Gamma'$ for $n + m = 10$. (For an explanation of how this table is read see **1**, p. 482).

	(10)	$(9,1)$	$(8,2)$	$(8,1^2)$	$(7,3)$	$(7,2,1)$	$(7,1^3)$	$(6,4)$	$(6,3,1)$	$(6,2^2)$	$(6,2,1^2)$	$(6,1^4)$	(5^2)	$(5,4,1)$	$(5,3,2)$	$(5,3,1^2)$	$(5,2^2,1)$	$(5,2,1^3)$	$(5,1^5)$	$(4^2,2)$	$(4^2,1^2)$	$(4,3^2)$	$(4,3,2,1)$	$(4,3,1^3)$	$(3^3,1)$	
$(9).(1)$	1	1																								$(1^9).(1)$
$(8,1).(1)$		1	1	1																						$(2,1^7).(1)$
$(7,2).(1)$			1		1	1																				$(2^2,1^5).(1)$
$(7,1^2).(1)$				1		1	1																			$(3,1^6).(1)$
$(6,3).(1)$					1			1	1																	$(2^3,1^3).(1)$
$(6,2,1).(1)$						1			1	1	1															$(3,2,1^4).(1)$
$(6,1^3).(1)$							1				1	1														$(4,1^5).(1)$
$(5,4).(1)$								1					1	1												$(2^4,1).(1)$
$(5,3,1).(1)$									1				1	1	1											$(3,2^2,1^2).(1)$
$(5,2^2).(1)$										1					1		1									$(3^2,1^3).(1)$
$(5,2,1^2).(1)$											1					1	1	1								$(4,2,1^3).(1)$
$(5,1^4).(1)$												1						1	1							$(5,1^4).(1)$
$(4^2,1).(1)$														1						1	1					$(3,2^3).(1)$
$(4,3,2).(1)$															1					1		1	1			$(3^2,2,1).(1)$
$(4,3,1^2).(1)$																1					1		1	1		$(4,2^2,1).(1)$
$(3^3).(1)$																						1			1	$(3^3).(1)$
$(8).(2)$	1	1	1																							$(1^8).(1^2)$
$(7,1).(2)$		1	1	1	1	1																				$(2,1^6).(1^2)$
$(6,2).(2)$			1		1	1		1	1	1																$(2^2,1^4).(1^2)$
$(6,1^2).(2)$				1		1	1		1		1															$(3,1^5).(1^2)$
$(5,3).(2)$					1			1	1				1	1	1											$(2^3,1^2).(1^2)$
$(5,2,1).(2)$						1			1	1	1			1	1	1	1									$(3,2,1^3).(1^2)$
$(5,1^3).(2)$							1				1	1				1		1								$(4,1^4).(1^2)$
$(4^2).(2)$								1						1						1						$(2^4).(1^2)$
$(4,3,1).(2)$									1					1	1	1				1	1	1	1			$(3,2^2,1).(1^2)$
	(1^{10})	$(2,1^8)$	$(2^2,1^6)$	$(3,1^7)$	$(2^3,1^4)$	$(3,2,1^5)$	$(4,1^6)$	$(2^4,1^2)$	$(3,2^2,1^3)$	$(3^2,1^4)$	$(4,2,1^4)$	$(5,1^5)$	(2^5)	$(3,2^3,1)$	$(3^2,2,1^2)$	$(4,2^2,1^2)$	$(4,3,1^3)$	$(5,2,1^3)$	$(6,1^4)$	$(3^2,2^2)$	$(4,2^3)$	$(3^3,1)$	$(4,3,2,1)$	$(5,2^2,1)$	$(4,3^2)$	

	(10)	$(9,1)$	$(8,2)$	$(8,1^2)$	$(7,3)$	$(7,2,1)$	$(7,1^3)$	$(6,4)$	$(6,3,1)$	$(6,2^2)$	$(6,2,1^2)$	$(6,1^4)$	(5^2)	$(5,4,1)$	$(5,3,2)$	$(5,3,1^2)$	$(5,2^2,1)$	$(5,2,1^3)$	$(5,1^5)$	$(4^2,2)$
$(4,2^2).(2)$										1					1		1			1
$(4,2,1^2).(2)$											1					1	1	1		
$(4,1^4).(2)$												1						1	1	
$(3^2,2).(2)$															1					
$(3^2,1^2).(2)$																1				
$(3,2^2,1).(2)$																	1			
$(3,2,1^3).(2)$																		1		
$(3,1^5).(2)$																			1	
$(2^4).(2)$																				
$(2^3,1^2).(2)$																				
$(2^2,1^4).(2)$																				
$(2,1^6).(2)$																				
$(1^8).(2)$																				
$(7).(3)$	1	1	1		1															
$(6,1).(3)$		1	1	1	1	1		1	1											
$(5,2).(3)$			1		1	1		1	1	1			1	1	1					
$(5,1^2).(3)$				1		1	1		1		1			1		1				
$(4,3).(3)$					1			1	1					1	1					1
$(4,2,1).(3)$						1			1	1	1			1	1	1	1			1
$(4,1^3).(3)$							1				1	1				1		1		
$(3^2,1).(3)$									1						1	1				
$(3,2^2).(3)$										1					1		1			
$(3,2,1^2).(3)$											1					1	1	1		
$(3,1^4).(3)$												1						1	1	
$(2^3,1).(3)$																	1			
$(2^2,1^3).(3)$																		1		
$(2,1^5).(3)$																			1	
$(1^7).(3)$																				
$(7).(2,1)$		1	1	1		1														
$(6,1).(2,1)$			1	1	1	2	1		1	1	1									
$(5,2).(2,1)$					1	1		1	2	1	1			1	1	1	1			
$(5,1^2).(2,1)$						1	1		1	1	2	1			1	1	1	1		
$(4,3).(2,1)$								1	1				1	2	1	1				1
$(4,2,1).(2,1)$									1	1	1			1	2	2	2	1		1
$(4,1^3).(2,1)$											1	1				1	1	2	1	
$(3^2,1).(2,1)$														1	1	1				1
$(6).(4)$	1	1	1		1			1												
$(5,1).(4)$		1	1	1	1	1		1	1				1	1						
$(4,2).(4)$			1		1	1		1	1	1				1	1					1
$(4,1^2).(4)$				1		1	1		1		1			1		1				
$(3^2).(4)$					1				1						1					
$(3,2,1).(4)$						1			1	1	1				1	1	1			
$(3,1^3).(4)$							1				1	1				1		1		
$(2^3).(4)$										1							1			
$(2^2,1^2).(4)$											1						1	1		
$(2,1^4).(4)$												1						1	1	
$(1^6).(4)$																			1	
$(6).(3,1)$		1	1	1	1	1			1											
$(5,1).(3,1)$			1	1	1	2	1	1	2	1	1			1	1	1				
$(4,2).(3,1)$					1	1		1	2	1	1		1	2	2	1	1			1
$(4,1^2).(3,1)$						1	1		1	1	2	1		1	1	2	1	1		1
$(3^2).(3,1)$								1	1					1	1	1				1
$(3,2,1).(3,1)$									1	1	1			1	2	2	2	1		1
$(3,1^3).(3,1)$											1	1				1	1	2	1	
$(2^3).(3,1)$															1		1			
	(1^{10})	$(2,1^8)$	$(2^2,1^6)$	$(3,1^7)$	$(2^3,1^4)$	$(3,2,1^5)$	$(4,1^6)$	$(2^4,1^2)$	$(3,2^2,1^3)$	$(3^2,1^4)$	$(4,2,1^4)$	$(5,1^5)$	(2^5)	$(3,2^3,1)$	$(3^2,2,1^2)$	$(4,2^2,1^2)$	$(4,3,1^3)$	$(5,2,1^3)$	$(6,1^4)$	$(3^2,2^2)$

	$(4,3,2,1)$ / $(4,3,2,1)$	$(4,3,1^3)$ / $(5,2^2,1)$	$(4,2^3)$ / $(4^2,1^2)$	$(4,2^2,1^2)$ / $(5,3,1^2)$	$(4,2,1^4)$ / $(6,2,1^2)$	$(4,1^6)$ / $(7,1^3)$	$(3^3,1)$ / $(4,3^2)$	$(3^2,2^2)$ / $(4^2,2)$	$(3^2,2,1^2)$ / $(5,3,2)$	$(3^2,1^4)$ / $(6,2^2)$	$(3,2^3,1)$ / $(5,4,1)$	$(3,2^2,1^3)$ / $(6,3,1)$	$(3,2,1^5)$ / $(7,2,1)$	$(3,1^7)$ / $(8,1^2)$	(2^5) / (5^2)	$(2^4,1^2)$ / $(6,4)$	$(2^3,1^4)$ / $(7,3)$	$(2^2,1^6)$ / $(8,2)$	$(2,1^8)$ / $(9,1)$	(1^{10}) / (10)
$(3^2,1^2).(1^2)$	1		1																	
$(4,2,1^2).(1^2)$	1	1		1																
$(5,1^3).(1^2)$		1			1															
$(3^2,2).(1^2)$	1						1	1												
$(4,2^2).(1^2)$	1	1					1		1											
$(4,3,1).(1^2)$	1		1	1				1	1		1									
$(5,2,1).(1^2)$		1		1	1				1	1		1								
$(6,1^2).(1^2)$					1	1				1			1							
$(4^2).(1^2)$			1								1				1					
$(5,3).(1^2)$				1							1	1				1				
$(6,2).(1^2)$					1							1	1				1			
$(7,1).(1^2)$						1							1	1				1		
$(8).(1^2)$														1					1	
$(1^7).(1^3)$																				
$(2,1^5).(1^3)$																				
$(2^2,1^3).(1^3)$																				
$(3,1^4).(1^3)$																				
$(2^3,1).(1^3)$																				
$(3,2,1^2).(1^3)$	1																			
$(4,1^3).(1^3)$		1																		
$(3,2^2).(1^3)$	1						1													
$(3^2,1).(1^3)$	1		1					1												
$(4,2,1).(1^3)$	1	1		1					1											
$(5,1^2).(1^3)$		1			1					1										
$(4,3).(1^3)$			1	1							1									
$(5,2).(1^3)$				1	1							1								
$(6,1).(1^3)$					1	1							1							
$(7).(1^3)$						1								1						
$(1^7).(2,1)$																				
$(2,1^5).(2,1)$																				
$(2^2,1^3).(2,1)$																				
$(3,1^4).(2,1)$																				
$(2^3,1).(2,1)$	1																			
$(3,2,1^2).(2,1)$	2	1	1	1																
$(4,1^3).(2,1)$	1	1		1	1															
$(3,2^2):(2,1)$	2	1					1	1	1											
$(1^6).(1^4)$																				
$(2,1^4).(1^4)$																				
$(2^2,1^2).(1^4)$																				
$(3,1^3).(1^4)$																				
$(2^3).(1^4)$																				
$(3,2,1).(1^4)$	1																			
$(4,1^2).(1^4)$		1																		
$(3^2).(1^4)$			1																	
$(4,2).(1^4)$				1																
$(5,1).(1^4)$					1															
$(6).(1^4)$						1														
$(1^6).(2,1^2)$																				
$(2,1^4).(2,1^2)$																				
$(2^2,1^2).(2,1^2)$	1																			
$(3,1^3).(2,1^2)$	1	1																		
$(2^3).(2,1^2)$	1						1													
$(3,2,1).(2,1^2)$	3	1	1	1			1	1	1											
$(4,1^2).(2,1^2)$	1	2		1	1				1	1										
$(3^2).(2,1^2)$	1		1	1				1			1									

	(10)	$(9,1)$	$(8,2)$	$(8,1^2)$	$(7,3)$	$(7,2,1)$	$(7,1^3)$	$(6,4)$	$(6,3,1)$	$(6,2^2)$	$(6,2,1^2)$	$(6,1^4)$	(5^2)	$(5,4,1)$	$(5,3,2)$	$(5,3,1^2)$	$5,2^2,1)$	$(5,2,1^3)$	$(5,1^5)$	$(4^2,2)$
$(2^2,1^2).(3,1)$																1	1	1		
$(2,1^4).(3,1)$																		1	1	
$(1^6).(3,1)$																				
$(6).(2^2)$			1			1				1										
$(5,1).(2^2)$					1	1			1	1	1				1		1			
$(4,2).(2^2)$								1	1	1				1	1	1	1			1
$(4,1^2).(2^2)$									1		1				1	1	1	1		
$(3^2).(2^2)$													1	1	1					
$(3,2,1).(2^2)$														1	1	1	1			1
$(5).(5)$	1	1	1		1			1					1							
$(4,1).(5)$		1	1	1	1	1		1	1					1						
$(3,2).(5)$			1		1	1			1	1					1					
$(3,1^2).(5)$				1		1	1		1		1					1				
$(2^2,1).(5)$						1				1	1						1			
$(2,1^3).(5)$							1				1	1						1		
$(1^5).(5)$												1							1	
$(4,1).(4,1)$			1	1	1	2	1	1	2	1	1		1	2	1	1				1
$(3,2).(4,1)$					1	1		1	2	1	1			1	2	1	1			1
$(3,1^2).(4,1)$						1	1		1	1	2	1		1	1	2	1	1		
$(2^2,1).(4,1)$									1	1	1				1	1	2	1		
$(2,1^3).(4,1)$											1	1				1	1	2	1	
$(3,2).(3,2)$								1	1	1			1	2	2	1	1			1
$(3,1^2).(3,2)$									1		1			1	1	2	1	1		1
$(2^2,1).(3.2)$														1	1	1	1			1
$(3,1^2).(3,1^2)$										1	1	1			1	1	2	2	1	1
	(1^{10})	$(2,1^8)$	$(2^2,1^6)$	$(3,1^7)$	$(2^3,1^4)$	$(3,2,1^5)$	$(4,1^6)$	$(2^4,1^2)$	$(3,2^2,1^3)$	$(3^2,1^4)$	$(4,2,1^4)$	$(5,1^5)$	(2^5)	$(3,2^3,1)$	$(3^2,2,1^2)$	$(4,2^2,1^2)$	$(4,3,1^3)$	$(5,2,1^3)$	$(6,1^4)$	$(3^2,2^2)$

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$(4,3,2,1)$	$(4,3,1^3)$	$(4,2^3)$	$(4,2^2,1^2)$	$(4,2,1^4)$	$(4,1^6)$	$(3^3,1)$	$(3^2,2^2)$	$(3^2,2,1^2)$	$(3^2,1^4)$	$(3,2^3,1)$	$(3,2^2,1^3)$	$(3,2,1^5)$	$(3,1^7)$	(2^5)	$(2^4,1^2)$	$(2^3,1^4)$	$(2^2,1^6)$	$(2,1^8)$	(1^{10})	
1	1	1	2	1				1		1	1									$(4,2).(2,1^2)$
	1		1	2	1				1		1	1								$(5,1).(2,1^2)$
				1	1							1	1							$(6).(2,1^2)$
																				$(1^6).(2^2)$
																				$(2,1^4).(2^2)$
1		1																		$(2^2,1^2).(2^2)$
1			1																	$(3,1^3).(2^2)$
1							1													$(2^3).(2^2)$
2	1	1	1			1	1	1		1										$(3,2,1).(2^2)$
																				$(1^5).(1^5)$
																				$(2,1^3).(1^5)$
																				$(2^2,1).(1^5)$
																				$(3,1^2).(1^5)$
																				$(3,2).(1^5)$
																				$(4,1).(1^5)$
																				$(5).(1^5)$
																				$(2,1^3).(2,1^3)$
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1		1	1																	$(3,2).(2,1^3)$
	1		1	1																$(4,1).(2,1^3)$
2		1				1	1													$(2^2,1).(2^2,1)$
2	1		1			1		1												$(3,1^2).(2^2,1)$
2	1	1	1				1	1		1										$(3,2).(2^2,1)$
2	2	1	1	1			1	1	1											$(3,1^2).(3,1^2)$
$(4,3,2,1)$	$(5,2^2,1)$	$(4^2,1^2)$	$(5,3,1^2)$	$(6,2,1^2)$	$(7,1^3)$	$(4,3^2)$	$(4^2,2)$	$(5,3,2)$	$(6,2^2)$	$(5,4,1)$	$(6,3,1)$	$(7,2,1)$	$(8,1^2)$	(5^2)	$(6,4)$	$(7,3)$	$(8,2)$	$(9,1)$	(10)	

THE JOHNS HOPKINS UNIVERSITY.

ON A CLASS OF ARITHMETICAL FOURIER SERIES.*

By PHILIP HARTMAN.

Let

$$\psi(x) = -\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{\sin 2k\pi x}{k} = \begin{cases} x-[x]-\frac{1}{2} & \text{if } x \neq [x] \\ 0 & \text{if } x = [x] \end{cases}, \tag{1}$$

then we have the formal identity

$$\sum_{n=1}^{\infty}\frac{c_n\psi(nx)}{n} = -\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\left(\sum_{d/k} c_d\right)\sin 2k\pi x. \tag{2}$$

There have been several papers justifying this process for particular sequences $\{c_n\}$; see, for example, Landau [1] for $c_n = n^{-1}$; Chowla [2] for $c_n = n^{\alpha}$, $\alpha < 0$; Chowla and Walfisz [3] for $c_n = 1$; Hartman and Wintner [4] for $c_n = n^{\alpha}$, $\alpha < \frac{1}{2}$; Davenport [5] for $c_n = \mu(n)$, for $c_n = \lambda(n)$, for $c_n = \Lambda(n)$, and, finally, for $c_{n^2} = \mu(n)$ and all other $c_n = 0$. In most of the above examples, (2) is shown to hold almost everywhere, but (2) is valid for all x in the cases treated by Landau,[1] Chowla,[2] and the last example of Davenport.[5]

A more general problem is treated in this paper. Let $f(x)$ be a function of period 1 which can be represented almost everywhere by a convergent trigonometrical series, say

$$f(x) = \tfrac{1}{2}a_0 + \sum_{k=1}(a_k\cos 2k\pi x + b_k\sin 2k\pi x). \tag{3}$$

Conditions on $f(x)$ and on sequences $\{c_n\}$ will be investigated under which the following identity

$$\sum_{n=1}^{\infty} c_n f(nx) = \tfrac{1}{2}a_0\sum_{k=1}^{\infty}c_k + \sum_{k=1}^{\infty}\left[\left(\sum_{d/k}c_d a_{k/d}\right)\cos 2k\pi x + \left(\sum_{d/k}c_d b_{k/d}\right)\sin 2k\pi x\right] \tag{4}$$

* Received July 28, 1937.

[1] E. Landau, "Konvergenzbeweis einer Lerchschen Reihe," *Mémoires de la Société Royale des Sciences de Bohême, Classe des Sciences*, (1919), IV.

[2] S. D. Chowla, "Some problems in diophantine approximation (I)," *Mathematische Zeitschrift*, vol. 33 (1931), pp. 544-563.

[3] S. D. Chowla and A. Walfisz, "Über eine Riemannsche Identität," *Acta Arithmetica*, vol. 1 (1935), pp. 87-112.

[4] P. Hartman and A. Wintner, "On certain Fourier series involving sums of divisors," to appear in the *Acta Arithmetica*.

[5] H. Davenport, "On some infinite series involving arithmetical functions," *Quarterly Journal of Mathematics*, vol. 8 (1937), pp. 8-13.

is valid. Landau, Chowla, Walfisz, and Davenport did not seem to recognize that the problem was a Fourier series problem and obtained their results with the aid of diophantine approximation. Fourier series methods are employed here; the methods are essentially the same as those used by Wintner.[6]

THEOREM 1. *Let $f(x)$ be a Lebesgue integrable function with the period 1 and*

$$f(x) \sim \sum_{k=1}^{\infty} (a_k \cos 2k\pi x + b_k \sin 2k\pi x). \tag{5}$$

Let α and β be two real numbers such that

$$m = \min(\alpha, \beta) > \tfrac{1}{2}, \tag{6}$$

and

$$a_k = O(k^{-\alpha}), \qquad b_k = O(k^{-\alpha}), \tag{7}$$

and, finally,

$$c_n = O(n^{-\beta}). \tag{8}$$

Then

(i) *the partial sums of the series*

$$\sum_{n=1}^{\infty} c_n f(nx) \tag{9}$$

tend in the mean (L^q) to a function $F(x)$, for every $q < 1/(1-m)$ or for every $q > 0$ according as $m < 1$ or $m \geqq 1$;

(ii) *$F(x)$ possesses the Fourier series*

$$F(x) \sim \sum_{k=1}^{\infty} \left[\left(\sum_{d/k} c_d a_{k/d}\right) \cos 2k\pi x + \left(\sum_{d/k} c_d b_{k/d}\right) \sin 2k\pi x\right]; \tag{10}$$

(iii) *the Fourier series of $F(x)$ converges for almost all x to $F(x)$,*

$$F(x) = \sum_{k=1}^{\infty} \left[\left(\sum_{d/k} c_d a_{k/d}\right) \cos 2k\pi x + \left(\sum_{d/k} c_d b_{k/d}\right) \sin 2k\pi x\right]. \tag{11}$$

It follows from the formal identity (4) that it was necessary to suppose $a_0 = 0$, unless $\sum_{n=1}^{\infty} c_n$ is a convergent series. The conditions imposed on $f(x)$ allow several immediate conclusions to be drawn. Suppose, first, that $\alpha < 1$ and let p be a number such that

$$2 > p > 1/\alpha,$$

[6] A. Wintner, "On a trigonometrical series of Riemann," *American Journal of Mathematics*, vol. 59 (1937), pp. 629-634.

so that

$$\sum_{k=1}^{\infty} (| a_k |^p + | b_k |^p) < \infty ;$$

then by the Hausdorff [7] extension of the Fischer-Riesz theorem, $f(x)$ belongs to the class $L^{p'}$, $p' = p/(p-1) < 1/(1-\alpha)$. By the same arguments, it follows that if $\alpha \geqq 1$, $f(x)$ belongs to L^q for all $q > 0$. Thus, in any case, $f(x)$ belongs to L^2. Also there exists a positive number δ such that

$$2\alpha - \delta > 1,$$

hence

$$\sum_{k=1}^{\infty} (a_k^2 + b_k^2) k^{\delta} < \infty,$$

this implies that the Fourier series (5) is convergent almost everywhere to $f(x)$,

$$f(x) = \sum_{k=1}^{\infty} (a_k \cos 2k\pi x + b_k \sin 2k\pi x). \tag{12}$$

Proof of (*i*). Suppose first that $m < 1$, it is to be proved that if

$$F_N(x) = \sum_{n=1}^{N} c_n f(nx), \tag{13}$$

then there exists a function $F(x)$ such that

$$\int_0^1 [F(x) - F_N(x)]^q dx \to 0 \text{ as } N \to \infty \text{ if } q < 1/(1-m). \tag{14}$$

Now it is clear that $F_N(x)$ is integrable and

$$F_N(x) \sim \sum_{k=1}^{\infty} \left[\left(\sum_{d/k}^{d \leqq N} c_d a_{k/d} \right) \cos 2k\pi x + \left(\sum_{d/k}^{d \leqq N} c_d b_{k/d} \right) \sin 2k\pi x \right]. \tag{15}$$

Let $M > N$, then it follows from the Hausdorff [7] extension of the Parseval relation that

$$\begin{aligned} \int_0^1 [F_M(x) - F_N(x)]^q dx \\ \leqq \left[\sum_{k=1}^{\infty} \left| \sum_{d/k}^{N < d \leqq M} c_d a_{k/d} \right|^{q/(q-1)} + \sum_{k=1}^{\infty} \left| \sum_{d/k}^{N < d \leqq M} c_d b_{k/d} \right|^{q/(q-1)} \right]^{q-1} \end{aligned} \tag{16}$$

provided that $q \geqq 2$, as one may suppose. Thus, if $q \geqq 2$,

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[7] F. Hausdorff, "Eine Ausdehnung des Parsevalschen Satzes über Fourierreihen," *Mathematische Zeitschrift*, vol. 16 (1923), pp. 163-169.

$$(17)\int_0^1 [F_M(x)-F_N(x)]^q dx \leqq \Big[\sum_{k=N}^{\infty}\Big(\sum_{d/k}|c_d a_{k/d}|\Big)^{q/(q-1)}+\sum_{k=N}^{\infty}\Big(\sum_{d/k}|c_d b_{k/d}|\Big)^{q/(q-1)}\Big]^{q-1}.$$

Now from (7) and (8)

$$(18)\qquad \sum_{d/k}|c_d a_{k/d}| = O\Big(\frac{1}{k^a}\sum_{d/k}d^{a-\beta}\Big)=O\Big(\frac{\sigma_{a-\beta}(k)}{k^a}\Big),$$

where

$$(19)\qquad \sigma_\gamma(k)=\sum_{d/k}d^\gamma.$$

It is known [8] that if $\gamma \geqq 0$, then

$$(20)\qquad \sigma_\gamma(k)=O(k^{\gamma+\epsilon}),$$

and if $\gamma \leqq 0$, then

$$(21)\qquad \sigma_\gamma(k)=O(k^{\epsilon}),$$

where $\epsilon>0$ is arbitrary. Thus from (6), (18), (20) and (21), it follows that

$$(22)\qquad \sum_{d/k}|c_d a_{k/d}| = O(k^{-m+\epsilon}),$$

and similarly

$$(23)\qquad \sum_{d/k}|c_d b_{k/d}| = O(k^{-m+\epsilon}).$$

Hence if $q<1/(1-m)$, then $q/(q-1)>1/m$, so that the series

$$(24)\qquad \sum_{k=1}^{\infty}\Big[\Big(\sum_{d/k}|c_d a_{k/d}|\Big)^{q/(q-1)}+\Big(\sum_{d/k}|c_d b_{k/d}|\Big)^{q/(q-1)}\Big]$$

is convergent. Thus, from (17),

$$(25)\qquad \int_0^1 [F_M(x)-F_N(x)]^q dx \to 0,\ \text{as } M, N\to\infty$$

if $q<1/(1-m)$. Thus, for the case $m<1$, (i) now follows from (25) and the completeness of the space L^q. The case where $m \geqq 1$ follows similarly.

Proof of (ii). This follows at once from (i), for if a_k^N, b_k^N are the k-th Fourier coefficients of $F_N(x)$, then it follows from (14) that

$$(26)\qquad \lim_{N\to\infty} a_k^N = A_k \quad\text{and}\quad \lim_{N\to\infty} b_k^N = B_k$$

exist and are the k-th Fourier coefficients of $F(x)$. Now from (15),

$$(27)\qquad a_k^N=\sum_{d/k}^{d\leqq N} c_d a_{k/d} \quad\text{and}\quad b_k^N=\sum_{d/k}^{d\leqq N} c_d b_{k/d},$$

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[8] T. H. Gronwall, "Some asymptotic expressions in the theory of numbers," *Transactions of the American Mathematical Society*, vol. 14 (1913), pp. 113-122.

so that (10) follows from (26) and (27).

Proof of (iii). From (10), (22), and (23), it follows that

$$A_k = O(k^{-m+\epsilon}) \quad \text{and} \quad B_k = O(k^{-m+\epsilon}), \tag{28}$$

so that there exists a $\delta > 0$ such that

$$\sum_{k=1}^{\infty} (A_k^2 + B_k^2) k^{\delta} < \infty, \tag{29}$$

for $\epsilon > 0$ and $\delta > 0$ can be chosen so that

$$2m - 2\epsilon - \delta > 1.$$

That the equality (11) is valid almost everywhere is a consequence of (10) and (29).

Next, there will be proved [9]

THEOREM 2. *If $f(x)$ satisfies the conditions of Theorem 1 when the inequality (6) is replaced by*

$$m > \tfrac{2}{3} \tag{30}$$

and if $f(x)$ is bounded, then the series (9) converges almost everywhere to $F(x)$.

Proof. Let p be a number satisfying

$$0 < p < 3, \tag{31}$$

then, in virtue of (30),

$$p(1 - m) < 1. \tag{32}$$

Also there exists an $\epsilon > 0$ and a number p satisfying (31), and a fortiori (32), such that

$$p(2m - 1 - \epsilon) > 1. \tag{33}$$

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[9] This theorem and a similar proof for the case where $f(x)$ is the function (1) and where $c_n = n^{-a}$, $a > 2/3$ was also known to Professor Walfisz (unpublished), as I understood after this paper was completed. Incidentally, a similar theorem was proved by F. Jerosch and H. Weyl, "Über die Konvergenz von Reihen, die nach periodischen Funktionen fortschreiten," *Mathematische Annalen*, vol. 66 (1909), pp. 67-80.

(Cf., also S. Chowla, "On some infinite series involving arithmetical functions" I, II, *Proceedings of the Indian Academy of Sciences*, Section A, vol. 5 (1937), pp. 511-516. These references were obtained from the *Zentralblatt für Mathematik*, vol. 17 (1937), p. 5 as the above journal was not available. Added Nov. 18, 1937.)

In view of (10) and (15)

$$\int_0^1 [F(x) - F_N(x)]^2 dx \leqq \sum_{k=1}^{\infty} \left[\left(\sum_{d/k}^{d>N} c_d a_{k/d} \right)^2 + \left(\sum_{d/k}^{d>N} c_d b_{k/d} \right)^2 \right],$$

so that from (22) and (23),

$$(34) \qquad \int_0^1 [F(x) - F_N(x)]^2 dx = O\Big(\sum_{k=N+1}^{\infty} k^{-2m+\epsilon} \Big) = O(N^{-2m+1+\epsilon}).$$

It is seen that if p satisfies (33), then

$$\sum_{N=1}^{\infty} \int_0^1 [F(x) - F_{[N^p]}(x)]^2 dx$$

is a convergent series; thus, it follows from Fatou's lemma that

$$(35) \qquad \sum_{N=1}^{\infty} [F(x) - F_{[N^p]}(x)]^2$$

is convergent almost everywhere. Hence for almost all x

$$(36) \qquad F_{[N^p]}(x) \to F(x) \text{ as } N \to \infty \text{ if } p \text{ satisfies (33).}$$

Now let j be an integer such that

$$(37) \qquad [N^p] \leqq j < [(N+1)^p],$$

then

$$F_j(x) - F_{[N^p]}(x) = \sum_{n=[N^p]}^{j} c_n f(nx) = O\Big(\sum_{n=[N^p]}^{[(N+1)^p]} n^{-m} \Big),$$

or

$$(38) \qquad F_j(x) - F_{[N^p]}(x) = O(N^{-pm} \cdot N^{p-1}) = O(N^{p(1-m)-1}).$$

Hence, if j satisfies (37),

$$(39) \qquad F_j(x) - F_{[N^p]}(x) \to 0 \text{ as } N \to \infty \text{ if } p \text{ satisfies (32).}$$

Thus (36), (39) and the existence of a p satisfying (32) and (33) implies the statement of Theorem 2.

From this theorem, the following result can be inferred: Let $f(x)$ be a bounded integrable function of period 1 such that

$$f(x) \sim a_0/2 + \sum_{k=1}^{\infty} (a_k \cos 2k\pi x + b_k \sin 2k\pi x)$$

and

$$a_k = O(k^{-(2/3)-\epsilon}), \qquad b_k = O(k^{-(2/3)-\epsilon}),$$

then for almost all x

$$S_N(x; f) = \sum_{n=1}^{N} f(nx) - N \int_0^1 f(t)\, dt = O(N^{2/3+\epsilon}).$$

The theorems just proved imply all of the earlier results mentioned at the beginning of the paper except those which replaced "almost all x" by "all x." To obtain such results, it is clear that the conditions on the sequence $\{c_n\}$ must be such as to insure the convergence of the series (9) everywhere, thus defining $F(x)$ by an everywhere convergent series instead of using a definition which leaves the function undetermined on a zero set, as in Theorem 1 or 2. It is also clear that $f(x)$ will have to satisfy more stringent conditions than those above. In this connection, one has

THEOREM 3. *Let the series*

$$\sum_{n=1}^{\infty} c_n \tag{40}$$

be absolutely convergent. Let $f(x)$ be a bounded integrable function of period 1 such that

$$g(u,x) = \frac{1}{u}\int_0^u [(f(x+t) + f(x-t) - 2f(x)]dt, \qquad u > 0, \tag{41}$$

is of bounded variation in any u-interval to the right of $u = 0$ (when considered as a function of u) for all x and such that

$$g(u,x) \to 0, \qquad u \to 0 \tag{42}$$

for all x. Let $V_n(t,x)$ denote the total variation of $g(nu, nx)$ in the interval $0 < u < t$ and suppose that

$$\sum_{n=1}^{\infty} |\, c_n \,|\, V_n(t,x) \tag{43}$$

is convergent for some $t = t(x) > 0$. The identity

$$\sum_{n=1}^{\infty} c_n f(nx) = \tfrac{1}{2}a_0 \sum_{n=1}^{\infty} c_n + \sum_{k=1}^{\infty}\left[\left(\sum_{d/k} c_d a_{k/d}\right)\cos 2k\pi x + \left(\sum_{d/k} c_d b_{k/d}\right)\sin 2k\pi x\right] \tag{44}$$

is then valid for all x.

Proof. This theorem is merely a rewriting of the de la Vallée Poussin test for the convergence of Fourier series for the case at hand. In fact, the conditions that $g(u,x)$ be of bounded variation in some interval $0 < u < a$ and that $g(u,x) \to 0$ as $u \to 0$ are precisely the conditions required by the de la Vallée Poussin test for the convergence of the Fourier series of $f(x)$ at the point x. Thus the conditions of the theorem imply that (3) is valid for all x.

That

$$(45)\quad F(x) = \sum_{n=1}^{\infty} c_n f(nx)$$

$$\sim \tfrac{1}{2}a_0 \sum_{k=1}^{\infty} c_n + \sum_{n=1}^{\infty}\left[\left(\sum_{d/k} c_d a_{k/d}\right)\cos 2k\pi x + \left(\sum_{d/k} c_d b_{k/d}\right)\sin 2k\pi x\right]$$

is an immediate consequence of the uniform convergence of the series (9), which, in turn, follows from the boundedness of $f(x)$ and the absolute convergence of the series (40).

One obtains from (41) that

$$g(nu, nx) = \frac{1}{nu}\int_0^{nu} [f(nx+t) + f(nx-t) - 2f(nx)]dt.$$

If the integration variable t is changed to nt, one has

$$(46)\qquad g(nu, nx) = \frac{1}{u}\int_0^{u} [f(nx+nt) + f(nx-nt) - 2f(nx)]dt.$$

Now put

$$(47)\qquad G(u, x) = \frac{1}{u}\int_0^{u} [F(x+t) + F(x-t) - 2F(x)]dt,$$

then it is clear from (46) that

$$(48)\qquad G(u, x) = \sum_{n=1}^{\infty} c_n g(nu, nx).$$

If $t = t(x) > 0$ is chosen so that the series (43) is convergent, $G(u, x)$ is of bounded variation in the interval $0 < u < t$, since the variation of the sum of functions does not exceed the sum of the variations of the functions. Also since $g(u, x)$ is a bounded function, the series in (48) is uniformly convergent, thus it follows from

$$g(nu, nx) \to 0, \qquad u \to 0,$$

that

$$(49)\qquad G(u, x) \to 0, \qquad u \to 0.$$

Thus the function $F(x)$ satisfies the criterion of de la Vallée Poussin at every point, hence (44) is valid at every point in virtue of (45). This completes the proof.

Theorem 3 implies the very particular cases treated by Landau, Chowla, and Davenport. Let the function $f(x)$ of Theorem 3 be the function (1) and let $g(u, x)$ have the corresponding meaning. Let n and x be fixed; suppose that x is not of the form $j/2n$, where j is any integer, otherwise $g(nu, nx) = 0$ for all u. Let

$$\frac{k-1}{n} < x < \frac{k}{n},$$

and put

$$\alpha_1 = \max\left(x - \frac{k-1}{n}, \frac{k}{n} - x\right), \qquad \alpha_2 = \frac{1}{n} - \alpha_1. \tag{50}$$

Then

$$\begin{aligned} &\psi(nx+nt) + \psi(nx-nt) - 2\psi(nx) = 0 \\ &\qquad \text{if } 0 < t < \alpha_2; \\ &\psi(nx+nt) + \psi(nx-nt) - 2\psi(nx) = (-1)^{\mu(x)} \\ &\qquad \text{if } r/n + \alpha_2 < t < r/n + \alpha_1; \qquad (r = 0, 1, \cdots), \\ &\psi(nx+nt) + \psi(nx-nt) - 2\psi(nx) = 0 \\ &\qquad \text{if } r/n + \alpha_1 < t < (r+1)/n + \alpha_2; \end{aligned} \tag{51}$$

where $\mu(x)$ is 0 or 1 according as $x < (2k-1)/2n$ or $x > (2k-1)/2n$. Hence

$$\begin{aligned} &g(nu, nx) = 0 \\ &\qquad \text{if } 0 < u < \alpha_2 \\ &g(nu, nx) = (-1)^{\mu(x)}[r(\alpha_1 - \alpha_2) + u - r/n - \alpha_2]/u \\ &\qquad \text{if } r/n + \alpha_2 < u < r/n + \alpha_1; \qquad (r = 0, 1, \cdots). \\ &g(nu, nx) = (-1)^{\mu(x)}(r+1)(\alpha_1 - \alpha_2)/u \\ &\qquad \text{if } r/n + \alpha_1 < u < (r+1)/n + \alpha_2; \end{aligned} \tag{52}$$

It follows from these relations, by straightforward appraisals which can be left to the reader, that

$$V_n(t, x) \leqq C \log nt, \quad \text{if} \quad t > \frac{1}{n} \tag{53}$$

where C is a constant independent of n, t and x. Also, it is obvious that (42) is satisfied. Thus, if the sequence $\{c_n\}$ is such that

$$\sum_{n=1}^{\infty} \left| \frac{c_n \log n}{n} \right| \tag{54}$$

is convergent, the identity (2) is valid for *all* x.

THE JOHNS HOPKINS UNIVERSITY.

THE STRUCTURE OF LOCAL CLASS FIELD THEORY.*

By O. F. G. Schilling.†

The gist of local class field theory may be sought in the characterization of the finite abelian extensions over perfect fields whose valuation groups are discrete and archimedian and whose residue class fields are finite Galois fields. M. Moriya and the author of the present paper recently investigated the generalization of this theory to infinite extensions of $\mathfrak{p}$-adic number fields.[1] We found that the structural theory of finite abelian extensions over such infinite fields is in general determined by the nature of the value group and the field of residue classes. The results obtained can be interpreted from a more complex viewpoint as a theory of sufficient conditions which admit the development of a local class field theory securing the same theorems as in the finite case. Thus one is naturally lead to the following problem which we wish to investigate in this paper: *Is it possible to prove structural properties of a field which is perfect with respect to a discrete archimedian valuation if some set of standard theorems of local class field theory is valid for the given ground field?* Or, we ask for necessary conditions of theorems in local class field theory.

We shall see that most of the standard theorems imply that the residue field of the given perfect field is algebraically perfect [2] and that it possesses for each integer n exactly one cyclic extension of degree n. One easily observes that the set of possible residue class fields thus described is rather extensive, thus it is necessary to impose further conditions upon the given ground field if one wishes to characterize the $\mathfrak{p}$-adic number fields we investigated in the paper already mentioned. The simplest assumption we found was to postulate the validity of the given set of theorems for all perfect subfields of the given field.

In order to avoid unnecessary repetitions in our later investigations it is advisable to recall some known facts about the algebraic and arithmetical

* Received September 28, 1937.

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[1] M. Moriya and O. F. G. Schilling, "Zur Klassenkörpertheorie über unendlichen perfekten Körpern," *Journal of the Fac. of Sci. Hokk. Imp. Univ.*, ser. I, vol. 5 (Sapporo, Japan) 1937.

[2] We shall use "algebraically perfect" as translation of the German term "vollkommen."

theory of perfect fields. Let k be a field which is perfect with respect to a discrete archimedian valuation $\mathfrak{p}$. The maximal order of all $\mathfrak{p}$-adic integers in k be denoted by $\mathfrak{o}$, and let $\mathfrak{p} = (\pi)$ be the prime ideal of $\mathfrak{o}$. Then the ring $\mathfrak{o}/\mathfrak{p}$ of residue classes is a field $\boldsymbol{k}$. In case that $\boldsymbol{k}$ is an algebraically perfect field the structure of k is uniquely determined by that of $\boldsymbol{k}$ provided that the characteristic $\chi(\boldsymbol{k})$ is a unit or prime element of k.[3] There can be established a one to one correspondence between the finite algebraic extensions $\boldsymbol{U}$ of $\boldsymbol{k}$ and the unramified extensions U of k, the fields $\boldsymbol{U}$ being the residue class fields of the fields U. In particular, if U is a normal extension with the Galois group $\Gamma(U,k)$ then the Galois group $\Gamma(\boldsymbol{U},\boldsymbol{k})$ of the residue class field is equal to $\Gamma(U,k)$.

The arithmetic theory of normal division algebras D of finite rank m over k is very simple because of the validity of Hensel's criterion of reducibility.[4] All quantities of D which satisfy a minimal equation with highest coefficient one and whose coefficients lie in $\mathfrak{o}$ form a maximal order $\mathfrak{O}$, these elements are characterized by the property that their reduced norms are $\mathfrak{p}$-adic integers. The ring $\mathfrak{O}$ contains a uniquely determined principal prime ideal $\mathfrak{P}$ whose powers exhaust the set of all ideals with respect to $\mathfrak{O}$. The ring of residue classes $\mathfrak{O}/\mathfrak{P}$ is a division algebra of finite rank f over the field $\mathfrak{o}/\mathfrak{p}$, f being called the residue degree of D with respect to k.[5] Since $\mathfrak{O}$ is a principal ring the extension $\mathfrak{O}\mathfrak{p} = \mathfrak{p}\mathfrak{O}$ of the prime ideal $\mathfrak{p}$ is a positive power $\mathfrak{P}^e$ of the two-sided prime ideal $\mathfrak{P}$; the integer e is called the ramification degree of D with respect to k. Between the two numbers e and f related to D with respect to k the following important relation holds

$$ef = m.$$

Since a prime element Π of $\mathfrak{P}$ generates a ramified subfield $k(\Pi)$ of D whose ramification exponent is equal to e we find that e is a divisor of the degree $n = m^{1/2}$ of D.[6] Hence

$$e \leqq n \quad \text{and} \quad f \geqq n.$$

A normal division algebra D of degree n over k can always be represented

[3] H. Hasse and F. K. Schmidt, "Die Struktur diskret bewerteter Körper," *Crelle*, vol. 170 (1933).

[4] H. Hasse, "Über p-adische Schiefkörper und ihre Bedeutung für die Arithmetik hyperkomplexer Zahlsysteme," *Mathematische Annalen*, vol. 104 (1931).

[5] M. Deuring, "Algebren," *Ergebnisse der Mathematik und ihrer Grenzgebiete*, Berlin 1935, Chap. VI, §§ 11, 12.

[6] T. Nakayama, "Divisionalgebren über diskret bewerteten perfekten Körpern," *Crelle*, vol. 177, 1937.

in its class of normal algebras over k as the crossed product $(a(\sigma,\tau), U, \Gamma(U,k))$ belonging to a suitably chosen normal unramified splitting field U of D. For an arbitrary but fixed choice of a prime element π of $\mathfrak{p}$ this crossed product decomposes uniquely into a cyclic ramified algebra and an unramified algebra

$$(*) \qquad D \sim (a(\sigma,\tau), U, \Gamma(U,k)) \sim (\pi^{e(\sigma,\tau)}, U, \Gamma(U,k)) \times (\eta(\sigma,\tau), U, \Gamma(U,k)),$$

where $a(\sigma,\tau) = \pi^{e(\sigma,\tau)}\eta(\sigma,\tau)\epsilon(\sigma,\tau)$; $\eta(\sigma,\tau)$ belonging to a fixed set of multiplicative representatives of the residue class field $\boldsymbol{U}$ and $\epsilon(\sigma,\tau) \equiv 1 \pmod{\mathfrak{p}}$. Since the exponents $e(\sigma,\tau)$ form a set of addends belonging to the Galois group $\Gamma(U,k)$ they uniquely determine a cyclic subfield of U. It is readily seen that the algebra $(\pi^{e(\sigma,\tau)}, U, \Gamma(U,k))$ is similar to a ramified division algebra possessing the aforementioned cyclic subfield of U as splitting field. The second factor in (*) represents an unramified division algebra D' which corresponds to a division algebra $\boldsymbol{D}'$ over $\boldsymbol{k}$; we observe that $\boldsymbol{D}'$ need not be normal over $\boldsymbol{k}$.[7]

A simple group theoretical consideration shows that the group of all ramified algebras $\{(\pi^{e(\sigma,\tau)}, U, \Gamma(U,k))\}$ split by a fixed normal unramified field U over k is isomorphic with the factor group $\Gamma(U,k)/\Gamma(U,k)'$ of $\Gamma(U,k)$ with respect to its commutator group.

A normal division algebra D possesses always unramified maximally commutative subfields U provided that the algebra of residue classes $\mathfrak{O}/\mathfrak{P}$ has a center which is separable over the field $\boldsymbol{k}$.[8]

We begin our investigations with the proof of some lemmas which will be quite useful later on.

LEMMA 1. *If the ground field k and its finite algebraic extensions K possess no cyclic unramified extensions then there do not exist proper division algebras of finite rank over k,* i. e. *k is quasi-algebraically closed.*[9]

Proof. Suppose that D is a proper normal division algebra of rank $m = n^2$ over k. The algebra D possesses always separable normal splitting fields U:

$$D \times U \sim U.$$

[7] E. Witt, "Schiefkörper über diskret bewerteten Körpern," *Crelle*, vol. 176 (1937).

[8] Cf. the paper mentioned under [6]. In the following investigation we shall assume that all division algebras considered possess residue algebras with separable centers over $\boldsymbol{k}$.

[9] A field is called quasi-algebraically closed if there exist no proper division algebras over it.

Take such a splitting field and consider a Sylow subgroup Σ_q belonging to a prime divisor $q \neq 1$ of n. Let the subfield of U belonging to it be U_q, $([U_q : k], q) = 1$. Since Σ_q is a solvable group we can draw a composition chain

$$1 = \Sigma_q^{(s)} < \Sigma_q^{(s-1)} < \cdots < \Sigma_q^{(i)} < \Sigma_q^{(i-1)} < \cdots < \Sigma_q^{(2)} < \Sigma_q^{(1)} = \Sigma_q.$$

whose factors are cyclic groups of order q. The corresponding chain of fields may be given by

$$U = U_q^{(s)} > U_q^{(s-1)} > U_q^{(s-2)} > \cdots$$
$$> U_q^{(i)} > U_q^{(i-1)} > \cdots > U_q^{(2)} > U_q^{(1)} = U_q.$$

Now consider the algebra $D \times U_q^{(s-1)}$, it possesses the cyclic unramified field $U = U_q^{(s)}$ as splitting field. Since such a field cannot exist according to our assumptions we see that $D \times U_q^{(s-1)}$ must be similar to $U_q^{(s-1)}$. Repeating this conclusion we observe that already

$$D \times U_q \sim U_q.$$

This last relation contradicts the choice of q for a well known theorem in the algebraic theory of normal algebras asserts that the degree of a splitting field of D must be a multiple of n, and we have shown that D—if it would exist—possesses U_q as splitting field.

LEMMA 2. *If k and all its finite algebraic extensions are never centers of cyclic proper algebras then k is quasi-algebraically closed.*

Proof. We can apply again the same argument, here the relation $D \times U_q \sim U_q$ is a consequence of the supposed non-existence of cyclic algebras over U_q.

LEMMA 3. *If all possibly existing division algebras $\boldsymbol{D}$ of degree n over $\boldsymbol{k}$ and its finite algebraic extensions $\mathbf{K}$ possess isomorphic maximally commutative subfields and if $-\mathbf{1}$ is a universal quadratic norm in case that $\chi(\boldsymbol{k}) \neq 2$ then $\boldsymbol{k}$ is quasi-algebraically closed.*[10]

Proof. Suppose that there exists a proper normal division algebra $\boldsymbol{D}$ of degree $n = \Pi q_i^{\sigma_i}$ over $\boldsymbol{k}$. The algebra $\boldsymbol{D}$ is similar to a direct product of primary division algebras $\boldsymbol{D}_i$ whose degrees are equal to $q_i^{\sigma_i}$. In order to prove that

[10] An element is said to be a universal quadratic norm if it is the norm of a suitable element in any quadratic extension of the ground field.

our assumptions imply $\boldsymbol{D} \sim \boldsymbol{k}$ it suffices to show that all primary algebras $\boldsymbol{D}_i$ are similar to $\boldsymbol{k}$. First we consider division algebras whose degrees are relatively prime to 2.

Let $\boldsymbol{D} = (\boldsymbol{a}, \boldsymbol{Z}_r)$ be a cyclic division algebra of degree r. Then $\boldsymbol{a}^r$ is the least power of $\boldsymbol{a}$ which is norm of an element in $\boldsymbol{Z}_r$. Hence $\boldsymbol{k}(\boldsymbol{a}^{1/r})$ is an algebraic extension of $\boldsymbol{k}$ of degree r, since $\boldsymbol{a}$ is a factor set the field $\boldsymbol{k}(\boldsymbol{a}^{1/r})$ is a splitting field of $\boldsymbol{D}$. Our assumption yields that $\boldsymbol{Z}_r$ and $\boldsymbol{k}(\boldsymbol{a}^{1/r})$ are equal if taken in one and the same algebraically closed field over $\boldsymbol{k}$. But now the element $\boldsymbol{a}$ is the norm of a quantity in $\boldsymbol{Z}_r$ for r was supposed to be relatively prime to 2. Consequently $\boldsymbol{D}$ has to be similar to $\boldsymbol{k}$. Next we apply Lemma 2 which asserts that there do not exist proper division algebras over $\boldsymbol{k}$ as center of these primary algebras. Hence all primary algebras whose degrees are relatively prime to 2 are similar to $\boldsymbol{k}$.

If the degree of a primary algebra is a power of 2 we have to distinguish two different cases:

$$\text{(i)}\ \chi(\boldsymbol{k}) = 2 \quad \text{and} \quad \text{(ii)}\ \chi(\boldsymbol{k}) \neq 2.$$

Case (i). According to the theory of primary division algebras and their subalgebras it is sufficient to prove that there do not exist proper division algebras of quaternions over $\boldsymbol{k}$ and its finite algebraic extensions for our assumption is to hold over $\boldsymbol{k}$ and its extensions, and the argument using a Sylow subgroup belonging to the prime 2 of the Galois group of a normal splitting field applies here too.

Suppose now that there exists a proper division algebra of degree 2 over $\boldsymbol{k}$ as center. Such an algebra $\boldsymbol{Q}$ has obviously the form

$$\boldsymbol{Q} = (\boldsymbol{a}, \boldsymbol{k}(\boldsymbol{C}))$$

where $\boldsymbol{C}^2 - \boldsymbol{C} = \boldsymbol{b} \neq 0$, $\boldsymbol{u}^2 = \boldsymbol{a}$, $\boldsymbol{u}^{-1}\boldsymbol{C}\boldsymbol{u} = \boldsymbol{C} + \boldsymbol{1}$ with $\boldsymbol{a}, \boldsymbol{b} \neq 0$ in $\boldsymbol{k}$. Then our assumption implies that $\boldsymbol{k}(\boldsymbol{u}) = \boldsymbol{k}(\boldsymbol{C})$ in one and the same algebraic closure of $\boldsymbol{k}$. Such an equality is impossible for $\boldsymbol{k}(\boldsymbol{u})$ is an inseparable extension of $\boldsymbol{k}$ and $\boldsymbol{k}(\boldsymbol{C})$ is a separable extension.

Case (ii). It suffices again to show that there do not exist proper division algebras of quaternions over $\boldsymbol{k}$ and its finite algebraic extensions. Let $\boldsymbol{Q} = (\boldsymbol{a}, \boldsymbol{k}(\boldsymbol{b}^{1/2}))$ be such an algebra. Our assumptions imply first of all that $\boldsymbol{k}(\boldsymbol{a}^{1/2}) = \boldsymbol{k}(\boldsymbol{b}^{1/2})$, i. e. $\boldsymbol{b} = \boldsymbol{a}\boldsymbol{c}^2$ with $\boldsymbol{c}$ in $\boldsymbol{k}$. Hence $\boldsymbol{Q} = (\boldsymbol{a}, \boldsymbol{k}(\boldsymbol{a}^{1/2}))$. The norm of $\boldsymbol{a}^{1/2}$ is equal to $-\boldsymbol{a}$. But $\boldsymbol{a}$ itself is also a norm since our special assumption, $-\boldsymbol{1} = N\boldsymbol{d}$ with $\boldsymbol{d}$ in $\boldsymbol{k}(\boldsymbol{a}^{1/2})$, implies $\boldsymbol{a} = (-\boldsymbol{1})(-\boldsymbol{a}) = N(\boldsymbol{d}\boldsymbol{a}^{1/2})$.

Remark. If we omit in the assumptions that $-\boldsymbol{1}$ be a universal quadratic

norm then we see that there exists exactly one division algebra over $\boldsymbol{k}$, namely the algebra $(-1, \boldsymbol{k}(-1^{1/2}))$. For the existence of a division algebra $(\boldsymbol{a}, \boldsymbol{k}(\boldsymbol{a}^{1/2}))$ implies that $\boldsymbol{a}$ is not the norm of an element in $\boldsymbol{k}(\boldsymbol{a}^{1/2})$ hence $-\mathbf{1}$ is not a norm and a fortiori not a square. The algebra $(-\mathbf{1}, \boldsymbol{k}(\boldsymbol{a}^{1/2}))$ must be equivalent to $(\boldsymbol{a}, \boldsymbol{k}(\boldsymbol{a}^{1/2}))$ as the assumption on the isomorphism of the maximally commutative subfields implies, consequently

$$(\boldsymbol{a}, \boldsymbol{k}(\boldsymbol{a}^{1/2})) = (-\mathbf{1}, \boldsymbol{k}(-\mathbf{1}^{1/2})).$$

THEOREM 1. *If the field of residue classes* $\boldsymbol{k}$ *of the perfect field* k *is algebraically perfect the three following statements are equivalent:*

(i) *the ramification degree* e *of each normal division algebra* D *of degree* n *over* k *equals* n,

(ii) *the algebra of residue classes belonging to each normal division algebra* D *over* k *is a commutative field, and*

(iii) *the maximally commutative ununified subfields of each division algebra* D *over* k *are isomorphic and, in case of* $\chi(\boldsymbol{k}) \neq 2$, -1 *is a universal quadratic norm in* $\boldsymbol{k}$.

Proof. If $e = f = n$ for a division algebra D then

$$D \sim (\pi^{e(\sigma,\tau)}, U, \Gamma(U, k))(\eta(\sigma,\tau), U, \Gamma(U, k))$$
$$\sim (\pi^{e(\sigma,\tau)}, U, \Gamma(U, k)) \sim (\pi, Z_n)^{\nu},$$

where Z_n is a maximally commutative cyclic unramified subfield of D and ν is relatively prime to n. Moreover, the algebra has exponent n. The maximal order $\mathfrak{O}$ of D contains the maximal order $\mathfrak{O}(Z_n)$ of Z_n and

$$\mathfrak{O}/\mathfrak{P} \supseteq \mathfrak{O}(Z_n)/\mathfrak{P} \cong \mathfrak{O}(Z_n)/\mathfrak{p}.$$

Since $[\mathfrak{O}/\mathfrak{P} : \boldsymbol{k}] = n$ we have

$$\mathfrak{O}/\mathfrak{P} = \mathfrak{O}(Z_n)/\mathfrak{P},$$

i. e. the algebra of residue classes is a commutative field.

Conversely, let D be a division algebra over k which satisfies (ii). Since $\boldsymbol{k}$ was assumed to be algebraically perfect the algebra D can be represented as a generalized crossed product

$$D = (a(\alpha, \beta, \gamma), U)$$

where U is a suitable maximally commutative unramified subfield of D and α, β, γ are elements of the Galois group belonging to the normal field containing U. Now (ii) implies

$$\mathfrak{O}/\mathfrak{P} \supseteq \mathfrak{O}(U)/\mathfrak{P} \cong \mathfrak{O}(U)/\mathfrak{p}.$$

If the algebra $\mathfrak{O}/\mathfrak{P}$ has the degree n' over its center $\boldsymbol{K}$ and $\boldsymbol{K}$ has the rank n'' over $\boldsymbol{k}$ the following relation holds between the different ranks

$$f = [\mathfrak{O}/\mathfrak{P} : \boldsymbol{k}] = n'^2 n'' = n'' = n.$$

Hence (i) and (ii) are equivalent statements.

Obviously (i) and (ii) imply that all maximally commutative unramified subfields U_n of division algebras D of degree n are isomorphic. Hence $\boldsymbol{k}$ is quasi-algebraically closed and -1 is a universal quadratic norm in $\boldsymbol{k}$, thus (iii) holds. Conversely, (iii) implies according to Lemma 3 that $\boldsymbol{k}$ is quasi-algebraically closed. But this fact implies $e = f = n$. Thus (i) and (iii) are equivalent.

Now let k be a perfect field for which any one of the statements of Theorem 1 holds. We wish to show by an example that such an assumption does not necessarily imply that the well known theorems of the local class field theory hold.[11]

Let $\boldsymbol{k}$ be the field of all roots of unity over the field of all rational numbers. Then the field $k = \boldsymbol{k}\{t\}$ consisting of all formal power series $\sum_{i > -\infty}^{\infty} \boldsymbol{a}_i t^i$ in one variable t with coefficients $\boldsymbol{a}_i$ in $\boldsymbol{k}$ is a perfect field. The unramified extensions U of k are fields of formal power series in t whose coefficients are taken from a field which is isomorphic with the residue field $\boldsymbol{U}$ of U. Let U be a normal unramified field with the Galois group $\Gamma(U, k) = \{\sigma, \tau, \cdots\}$. Then the group $G(U)$ consisting of all normal algebras over k which are split by U is isomorphic with the factor group $\Gamma(U, k)/\Gamma(U, k)'$ of the Galois group $\Gamma(U, k)$ with respect to its commutator group for each algebra in $G(U)$ has the form

$$(\pi^{e(\sigma,\tau)}, U, \Gamma(U, k)),$$

where π denotes a fixed prime element of p. This is true because $\boldsymbol{k}$ is a quasi-algebraically closed field.[12] Moreover all division algebras D over k are cyclic.

Now let U be in particular a normal unramified extension of k whose Galois group is not solvable, such fields always exist. Namely, we have only to take a normal extension $\boldsymbol{U}'$ over the field of all rational numbers and to form the

[11] For general statements about perfect fields also in later considerations see W. Krull, "Allgemeine Bewertungstheorie," *Crelle*, vol. 167 (1931).

[12] H. Hasse, "Die Struktur der R. Brauerschen Algebrenklassengruppe über einem algebraischen Zahlkörper," *Mathematische Annalen*, vol. 107 (1933).

join $\boldsymbol{U} = \boldsymbol{U'k}$.[13] The group of algebras belonging to such a field U is equal to one.

If U is an abelian extension then $G(U)$ is isomorphic with the Galois group $\Gamma(U, k)$. In particular, the group $G(Z_n)$ belonging to a cyclic unramified extension Z_n consists of a single cycle $\mathrm{Z}(n)$ of order n.

Consider now the set of all division algebras D of degree and exponent n, these algebras can all be represented as cyclic crossed products

$$(\pi, Z_n)^\nu,$$

where the exponents ν are relatively prime to n. Observe that all algebras (a, Z_n) whose factor sets are units in k, are similar to k. Thus the cyclic ramified extension $k(\pi^{1/n})$ of degree n over k is the universal splitting field of all algebras of degree n

$$G(k(\pi^{1/n})) \supseteq \text{all } G(Z_n).$$

A simple argument using π-adic approximations yields that $G(k(\pi^{1/n}))$ is an infinite abelian group of type $(n, n, \cdots)$.

Furthermore, we see that the factor group of norm classes k^*/NU^* belonging to an unramified abelian extension of degree n is a cyclic group of order n whose generator is representable by t. In order to prove this assertion one has only to represent U as the join of a set of mutually distinct cyclic subfields and to determine the respective factor groups in a composition chain of U.

Combining these results we observe

(i) the group of classes of algebras possessing degree and exponent n is infinite,

(ii) the group of algebras $G(U)$ split by an unramified field is a cycle of order n if and only if U is a cyclic field of degree n,

(iii) the factor group of norm classes associated with an abelian unramified extension of degree n is a cycle of order n.

Consequently most of the standard theorems of local class field theory do not hold for fields k supposing only that the residue fields $\boldsymbol{k}$ are quasi-algebraically closed. Thus we see that it is necessary to impose further conditions upon k in order that the old theory can be re-established.

THEOREM 2. *If the residue class field* $\boldsymbol{k}$ *of the perfect field* k *is quasi-*

[13] Such fields are for example the fields whose group is an alternating group of more than 4 variables.

algebraically closed and if for each unramified normal extension U_n of k there exists at least one normal division algebra D_n of degree n over k which is split by U_n then all unramified extensions of k are cyclic.

Proof. The first assumption implies that the group of algebras split by U_n is isomorphic with the factor group $\Gamma(U_n, k)/\Gamma(U_n, k)'$. The other assumption $D_n \times U_n \sim U_n$ implies that $G(U_n)$ contains a proper division algebra of degree n. Hence $G(U_n)$ is a cyclic group of order n. Moreover, the structural theory of D_n yields that U_n must be a cyclic field. Thus we observe that all normal unramified fields U_n are cyclic, consequently all unramified fields are cyclic and there exists exactly one cyclic unramified extension Z_n of degree n over k. Or, the residue class field $\boldsymbol{k}$ possesses exactly one cyclic extension $\boldsymbol{Z}_n$ for each degree n.

Now it is very easy to prove that the local class field theory is valid in full extent for k if we assume furthermore that Theorem 3 holds also for all finite algebraic extensions K of k. According to the theory of C. Chevalley we can confine ourselves to show that each field K_n of degree n over k is splitting field of the cyclic group of all algebras of degree n for all unramified fields are cyclic.[14] Let $D = (\pi, Z_n, \sigma_n)$ be a generator of the group $G(Z_n)$. Then $D \times K_n \sim K_n$ for

$$\begin{aligned} D \times K_n &\sim (\pi, Z_nK_n/K_n, \Gamma(Z_nK_n, K_n)) \\ &\sim (\Pi^e, Z_nK_n/K_n, \sigma_e) \sim 1, \end{aligned}$$

where Π denotes a prime element of K_n and e, f denote the ramification and residue degrees of K_n respectively. Remark that $Z_n \cap K_n = Z_f$ is the inertial field of K_n as in the classical theory.[15]

THEOREM 2′. *If each extension K_n of degree n over k is splitting field of all algebras of degree n then there exists for each integer n exactly one cyclic unramified extension Z_n of degree n over k.*

Proof. For each integer n there exists at least one totally ramified field K_n, $\mathfrak{p} = \mathfrak{P}^n$ in K_n. Our assumption yields that K_n is splitting field of a division algebra D_n, hence the ramification degree e of D is equal to n. Consequently we can apply Theorem 2 together with Lemma 3 and see that the unramified extensions U_n of k are all cyclic.

THEOREM 3. *All unramified extensions of k are cyclic if the groups of*

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[14] C. Chevalley, "La théorie du symbole de restes normiques," *Crelle*, vol. 169 (1932).
[15] Cf. Theorem 0 in the paper of Chevalley.

algebras $G(U_n)$ which belong to the unramified extensions U_n of degree n over k are cyclic groups of order n and if this property holds also for the respective groups over the finite algebraic extensions of the groundfield k.

Proof. According to the Galois theory it suffices to show that there exists for each degree n exactly one field U_n. Suppose then that U_n' and U_n'' are two distinct unramified normal fields of degree n then the relative degree $\bar{n}$ of their join $U_n'U_n''$ is greater than n. Hence $G(U_n'U_n'')$ is a cyclic group of order $\bar{n}$ containing the two cyclic groups $G(U_n')$ and $G(U_n'')$ both of order n. Consequently $G(U_n')$ and $G(U_n'')$ coincide. Now let D be an arbitrary normal division algebra of degree n lying in $G(U_n') = G(U_n'')$. The algebra D possesses U_n' and U_n'' as maximally commutative subfields. Our assumption implies in particular that there exist cyclic unramified extensions of degree n over k, namely otherwise we would arrive at a contradiction to Lemma 1. Consequently U_n' and U_n'' must be equal to a cyclic field if they are considered in the same algebraic closure of k.

In the preceding theorems we imposed conditions on the finite algebraic extensions of k as well as on the normal division algebras over k. We now wish to investigate the properties of perfect fields k for which only certain postulates concerning the normal division algebras are assumed, the conditions imposed upon the field k will be of purely commutative type.

THEOREM 4. *If all classes of normal algebras which are representable by division algebras of degree n form a group of order n and if for each prime q there exists at least one cyclic unramified extension Z_q of degree q over k then the cyclic unramified extensions of degree n are unique.*

Proof. It suffices to prove that there exists at least one cyclic unramified extension Z_n over k for each degree n. Namely, if Z_n is such an unramified field then

$$\begin{aligned}[G(Z_n):1] = [k^*/NZ^*_n:1] &= [\{\pi\}/\{\pi\}^n:1][\epsilon/NE:1] \\ &= n[k^*/NZ^*_n:1],\end{aligned}$$

and since $G(Z_n)$ is contained in the group of all algebras of degree n which is supposed to have the order n, we see that

> $G(Z_n)$ is a cyclic group of order n which coincides with the group of all algebras of degree n.

This statement is true for all cyclic unramified fields of degree n, moreover each generating algebra of the group of algebras of degree n is totally

ramified and possesses all cyclic unramified fields of degree n as maximally commutative subfields. Hence these cyclic fields coincide in one and the same algebraic closure of k.

According to the Galois theory it is sufficient to show that there exist unramified cyclic fields Z_{q^ν} whose degrees are powers of a single prime q.

We distinguish two cases

$$\text{(i)}\quad q = \chi(\mathbf{k}) \quad \text{and} \quad \text{(ii)}\quad q \neq \chi(\mathbf{k}).$$

Case i. Since there exists at least one unramified cyclic field Z_q there exists at least one cyclic extension $\mathbf{Z}_q$ of degree q over the residue class field $\mathbf{k}$. According to the theory of cyclic extensions of degree q^ν over fields of characteristic q there consequently must exist such extensions $\mathbf{Z}_{q^\nu}$.[16] The corresponding perfect fields Z_{q^ν} are unramified and cyclic possessing the fields $\mathbf{Z}_{q^\nu}$ as residue class fields.[17]

Case ii. Again it is sufficient to consider the cyclic extensions of the residue class field $\mathbf{k}$. The field $\mathbf{k}$ contains a certain maximal q^N-th root of unity, where N is either a finite integer—then the q^{N+1}-st roots of unity do not lie in $\mathbf{k}$—or N is put equal to ∞ in a formal sense—then $\mathbf{k}$ contains all q^N-th roots of unity for arbitrarily great N. In the first case the $q^{N+\nu}$-th roots of unity determine cyclic extensions of degree q^ν for all such cyclotomic fields have a cyclic Galois group. In the second case our assumption yields too the existence of cyclic fields $\mathbf{Z}_{q^\nu}$ for any ν. Namely, it implies that the index $[\mathbf{k}^* : \mathbf{k}^{*q}]$ is divisible by q. Let $\boldsymbol{a}$ be a representative of $\mathbf{k}^*/\mathbf{k}^{*q}$ which is not equal to a q-th power, then the radicals $\boldsymbol{a}^{1/q^\nu}$ generate cyclic fields $\mathbf{Z}_{q^\nu} = \mathbf{k}(\boldsymbol{a}^{1/q^\nu})$ of degree q^ν.

Remark. The result of Theorem 4 can also be obtained if we substitute for the first assumption the following

> all classes of normal algebras over k possessing the exponent n form a set of n elements.

First we observe that all these classes form a group for if D_1 and D_2 are any two elements in the set, i. e. $D_1{}^n \sim D_2{}^n \sim k$, then

$$(D_1 \times D_2)^n \sim D_1{}^n \times D_2{}^n \sim k \quad \text{and} \quad (D_i{}^{-1})^n \sim (D_i{}^n)^{-1} \sim k.$$

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[16] E. Witt, "Zyklische Körper und Algebren der Characteristik p vom Grade p^n," *Crelle*, vol. 176 (1937).

[17] H. Hasse, "Die Gruppe der p^n-primären Zahlen für einen Primteiler $\mathfrak{p}$ von p," *Crelle*, vol. 176 (1937).

Furthermore this group of algebras is cyclic; in order to prove this assertion it is sufficient to show that all groups of algebras possessing exponents q^i where q is a prime are cyclic. But this is obvious since the group having exponent q^i contains the group having exponent q^{i-1} and since the factor group has the order q according to our assumption. Now we may reason as before, the group of algebras of exponent q^i contains in particular the ramified division algebras (π, Z_{q^i}) belonging to the cyclic unramified fields Z_{q^i}.

We wish to observe that the unique existence of the cyclic unramified fields Z_n for each degree n over k does not imply that these fields are the only existing unramified extensions over k. Namely consider the following example. Let $\boldsymbol{k}_0$ be the field of all roots of unity over the field of all rational numbers; let $\boldsymbol{a}' \neq 0$ be an arbitrary element of $\boldsymbol{k}_0$ then there exists a maximal integer $N(\boldsymbol{a}')$ such that $\boldsymbol{a}'^{1/N(\boldsymbol{a}')}$ lies in $\boldsymbol{k}_0$ but not $\boldsymbol{a}'^{1/N(\boldsymbol{a}')\nu}$ for $\nu > 1$. Put $\boldsymbol{a} = \boldsymbol{a}'^{1/N(\boldsymbol{a}')}$. Next adjoin to $\boldsymbol{k}_0$ all solutions of all solvable equations whose normal fields do not possess the fields $\boldsymbol{k}_0(\boldsymbol{a}^{1/n})$, $n > 1$, as subfields. Let the resulting enumerable algebraic extension of $\boldsymbol{k}_0$ be $\boldsymbol{k}_1$. The field $\boldsymbol{k}_1$ is also quasi-algebraically closed, moreover $\boldsymbol{k}_1(\boldsymbol{a}^{1/n})$ are cyclic extensions of degree n over $\boldsymbol{k}_1$. Adjoin to $\boldsymbol{k}_1$ again all solutions of all solvable equations in $\boldsymbol{k}_1$ which do not contain $\boldsymbol{a}^{1/n}$, etc. The field $\boldsymbol{k}$ which we obtain finally is quasi-algebraically closed and it does not contain $\boldsymbol{a}^{1/n}$, moreover there do exist non-solvable extensions of sufficiently high degrees over $\boldsymbol{k}$. The perfect field k be now the field of all formal power series in one variable t with coefficients in $\boldsymbol{k}$. Evidentally all normal division algebras of degree n over $\boldsymbol{k}$ are powers of the algebra $(t, \boldsymbol{k}(\boldsymbol{a}^{1/n})\{t\}/k)$. Hence they form a cyclic group of classes of algebras having order n. The assumptions of Theorem 4 are fulfilled by the field k, but there exist other unramified extensions U_n over k which are not cyclic. Thus we see that the statement of Theorem 4 is the best possible.

In the following investigations we shall be concerned with the implications for the perfect field k if we assume certain properties to hold for the factor group of norm classes.

LEMMA 4. *If the class group k^*/NZ^*_n is cyclic of order n for all unramified cyclic extensions of degree n and if the same is true for all unramified extensions U of k then the field of residue classes $\boldsymbol{k}$ is quasi-algebraically closed.*

Proof. The assumption $k^*/NZ^*_n \cong \mathrm{Z}(n)$ yields that

$$\boldsymbol{k}^*/N\boldsymbol{Z}^*_n = 1,$$

for

$$[k^*/NZ^*_n : 1] = [\{\pi\}/\{\pi\}^n : 1][\epsilon/N\mathrm{E} : 1] = n[\boldsymbol{k}^*/\boldsymbol{Z}^*_n : 1].$$

Moreover, we observe that the field of residue classes $\boldsymbol{k}$ is algebraically perfect. For otherwise there would exist proper cyclic and abelian division algebras $\boldsymbol{D}$ over $\boldsymbol{k}$ whose degrees are powers of the characteristic $\chi(\boldsymbol{k})$.[18] There would exist at least one cyclic unramified field Z_q over k whose field of residue classes is equal to a cyclic field $\boldsymbol{Z}_q$ which is splitting field of a normal division algebra over $\boldsymbol{k}$.[19] The norm class group k^*/NZ^*_q has then an order which is greater than q in contradiction to $k^*/NZ^*_q \cong \mathrm{Z}(q)$.

Next let $\boldsymbol{D}$ be an arbitrary normal division algebra over $\boldsymbol{k}$, we must show that $\boldsymbol{D} \sim \boldsymbol{k}$. The algebra $\boldsymbol{D}$ possesses at least one normal splitting field $\boldsymbol{U}$: $\boldsymbol{D} \sim (\boldsymbol{a}(\sigma,\tau), \boldsymbol{U}, \Gamma(\boldsymbol{U},\boldsymbol{k}))$. Let q be a prime divisor of the degree m of $\boldsymbol{D}$ which is supposed to be greater than 1. $\boldsymbol{\Sigma}_q$ be a Sylow subgroup of $\Gamma(\boldsymbol{U},\boldsymbol{k})$ belonging to the prime q, and let $\boldsymbol{U}_q$ be the corresponding subfield of $\boldsymbol{U}$, then $[\boldsymbol{U}_q : \boldsymbol{k}] \not\equiv 0 \pmod q$. Since $\boldsymbol{\Sigma}_q$ is solvable there exist at least one chain of fields between $\boldsymbol{U}_q$ and $\boldsymbol{U}$

$$\boldsymbol{k} \leq \boldsymbol{U}_q = \boldsymbol{U}_q{}^{(0)} \leq \boldsymbol{U}_q{}^{(1)} \leq \cdots \leq \boldsymbol{U}_q{}^{(i-1)} \leq \boldsymbol{U}_q{}^{(i)} \leq \cdots \leq \boldsymbol{U}_q{}^{(s-1)} \leq \boldsymbol{U}_q{}^{(s)} = \boldsymbol{U}$$

such that $\boldsymbol{U}_q{}^{(i)}$ are cyclic extensions of degree q over $\boldsymbol{U}_q{}^{(i-1)}$, $i = 1, 2, \cdots, s$. Consider now the algebra $\boldsymbol{D} \times \boldsymbol{U}_q$, it will be similar to $\boldsymbol{U}_q$ as the argument used in the preceding Lemma 2 and the relation $\boldsymbol{k}^*/N\boldsymbol{Z}^*_n = 1$ which is supposed to hold also for the finite extensions of $\boldsymbol{k}$, show. Again $\boldsymbol{D} \times \boldsymbol{U}_q \sim \boldsymbol{U}_q$ leads to a contradiction. Hence $\boldsymbol{k}$ is quasi-algebraically closed, moreover it is algebraically perfect for fields $\boldsymbol{k}$ which are not algebraically perfect are centers of normal division algebras.

THEOREM 5. *If $k^*/NA^* \cong \Gamma(A,k)$ holds for all unramified abelian extensions A of k then there exists for each degree n a uniquely determined cyclic field of degree n.*

Proof. Let A be an arbitrary unramified abelian field of degree n over k, then

$$[k^*/NA^* : 1] = [\{\pi\}/\{\pi\}^n : 1][\epsilon/N\mathrm{E} : 1].$$

Hence

$$\Gamma(A,k) \cong \mathrm{Z}(n) \quad \text{and} \quad \epsilon/N\mathrm{E} \cong \boldsymbol{k}^*/N\boldsymbol{A}^* = 1,$$

i. e. all unramified abelian fields are cyclic. It is then obvious that there exists for each degree n exactly one cyclic unramified extension Z_n of k.

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[18] A. A. Albert, "Normal division algebras over a modular field," *Transactions of the American Mathematical Society*, vol. 36 (1934); G. Köthe, "Über Schiefkörper mit Unterkörpern zweiter Art über dem Zentrum," *Crelle*, vol. 166 (1932).

[19] T. Nakayama, "Über die Algebren über einem Körper von der Primzahlcharakteristik, II," *Proceedings of the Imperial Academy of Tokyo*, vol. 12 (1937).

THEOREM 6. *If $k^*/NU^* \cong \Gamma(A, k)$ holds for all normal unramified extensions U over k where A denotes the maximal abelian subfield of U then there exist only cyclic unramified extensions Z_n over k.*

Proof. Let U_n be an arbitrary unramified normal extension of k then

$$[k^*/NU^*_n : 1] = [\{\pi\}/\{\pi\}^n : 1][\epsilon/N\mathrm{E} : 1] = n,$$

for

$$[k^*/NU^*_n : 1] = [k^*/NA^* : 1] \leqq n.$$

Consequently

$$\Gamma(A, k) \cong \mathrm{Z}(n),$$

hence

$$\Gamma(U_n, k) \cong \mathrm{Z}(n)$$

for $\Gamma(A, k)$ is a homomorphic map of $\Gamma(U_n, k)$.

Again as in Lemma 4 we see that $\boldsymbol{k}$ is algebraically perfect. Moreover, the field $\boldsymbol{k}$ is quasi-algebraically closed. Since $\boldsymbol{k}$ is algebraically perfect there do not exist normal division algebras of degree $\chi(\boldsymbol{k})^\nu$ over $\boldsymbol{k}$. Let then $\boldsymbol{D}$ be a normal division algebra over $\boldsymbol{k}$ whose degree is relatively prime to $\chi(\boldsymbol{k})$. Such an algebra is cyclic for $\boldsymbol{k}$ possesses only cyclic extensions $\boldsymbol{Z}_n$. The algebra $\boldsymbol{D}$ is the algebra of residue classes of a normal division algebra D over k belonging to $G(Z_n)$ where Z_n is the cyclic unramified extension of k corresponding to $\boldsymbol{Z}_n$. Since $G(Z_n)$ is a cyclic group of order n containing only ramified algebras the assumption $\boldsymbol{D} \not\sim \boldsymbol{k}$ leads to a contradiction.

LEMMA 5. *If $[k^*/NZ^*_n : 1] = n$ holds for all cyclic extensions of degree n over k then there exist cyclic unramified extensions of degree n with respect to k.*

Proof. We distinguish two cases

$$\text{(i)}\ \ (n, \chi(\boldsymbol{k})) = 1 \quad \text{and} \quad \text{(ii)}\ \ n = \chi(\boldsymbol{k})^\nu.$$

It is obviously sufficient to prove the assertion of the lemma for degrees n which are powers of a single prime q.

Case i. Suppose that there do not exist cyclic unramified extensions Z_{q^ν} of degree q^ν, then there do not exist cyclic extensions $\boldsymbol{Z}_{q^\nu}$ of degree q^ν over the field of residue classes $\boldsymbol{k}$. Hence the fields $\boldsymbol{k}$ and k respectively must contain all q^ν-th roots of unity. Moreover, $[\boldsymbol{k}^* : \boldsymbol{k}^{*q}] = [\boldsymbol{k}^* : \boldsymbol{k}^{*q^\nu}] = 1$. For if $[\boldsymbol{k}^* : \boldsymbol{k}^{*q}]$ were divisible by q then any representative $\boldsymbol{a}$ of $\boldsymbol{k}^*/\boldsymbol{k}^{*q}$ which is not equivalent to 1 mod $\boldsymbol{k}^{*q}$ would generate cyclic extensions $\boldsymbol{Z}_{q^\nu} = \boldsymbol{k}(\boldsymbol{a}^{1/q^\nu})$ of

degree q^ν over $\boldsymbol{k}$. Since $[k^*/NZ^*_n:1]=n$ for all unramified fields Z_n implies that $\boldsymbol{k}$ is algebraically perfect there is a one to one correspondence between the cyclic unramified extensions Z_{q^ν} over k and the cyclic extensions $\boldsymbol{Z}_{q^\nu}$ over $\boldsymbol{k}$, the field $\boldsymbol{Z}_{q^\nu}=\boldsymbol{k}(\boldsymbol{a}^{1/q^\nu})$ must be equal to $\boldsymbol{k}$.

Next $[\boldsymbol{k}^*:\boldsymbol{k}^{*q^\nu}]=1$ implies that all elements of $\boldsymbol{k}$ are q^ν-th powers. Consequently all units ϵ of k are q^ν-th powers as a simple p-adic approximation yields.

Now consider the cyclic ramified field $K_{q^\nu}=k(\pi^{1/q^\nu})=k((\epsilon\pi)^{1/q^\nu})$. The group of all norms of elements in K_{q^ν} contains the group of all units ϵ of k for they are all q^ν-th powers as we have just observed. Moreover $NK^*_{q^\nu}$ contains the prime element π, namely if $q\neq 2$ then $N(\pi^{1/q^\nu})=\pi$ and if $q=2$ then -1 is a square or 2^ν-th power in k consequently $\pi=-1\cdot-\pi$ is a norm. Hence $NK^*_{q^\nu}=k^*$ in contradiction to the assumption of the Lemma. Thus we find that k must possess cyclic unramified extensions Z_{q^ν}, they will be found among the cyclotomic extensions if ζ_{q^N} (N finite) is the maximal root of unity lying in k or they will be radical fields if all q^ν-th roots of unity lie in k.

Case ii. The assumption yields that there must exist proper division algebras D of degree q^ν over k, namely among possible other algebras the generators D_{q^ν} of the cyclic groups $G(Z'_{q^\nu})$ where Z'_{q^ν} denotes a ramified cyclic extension of degree q^ν over k. The existence of such fields has been established in the general theory of cyclic extensions over perfect fields whose residue class fields $\boldsymbol{k}$ are algebraically perfect.[20]

Now let D_{q^ν} be an arbitrary division algebra of degree q^ν over k, it possesses at least one unramified normal splitting field U for $\boldsymbol{k}$ is algebraically perfect:

$$D_{q^\nu}\sim(a(\sigma,\tau),U,\Gamma(U,k))\sim(\pi^{e(\sigma,\tau)},U,\Gamma(U,k))(\eta(\sigma,\tau),U,\Gamma(U,k)).$$

The algebra $(\eta(\sigma,\tau),U,\Gamma(U,k))$ which could be similar to an unramified algebra of degree q^μ $(\mu\leq\nu)$ must be similar to k for its residue algebra $(\eta(\sigma,\tau) \bmod \mathfrak{p},\boldsymbol{U},\Gamma(\boldsymbol{U},\boldsymbol{k}))$ is similar to $\boldsymbol{k}$, $\boldsymbol{k}$ being an algebraically perfect field.[21] Hence

$$D_{q^\nu}\sim(\pi^{e(\sigma,\tau)},U,\Gamma(U,k))\sim(\pi,Z_{q^\rho})^\sigma,$$

where Z_{q^ρ} denotes the unramified cyclic subfield of U which corresponds to the character of $\Gamma(U,k)$ induced by the set of addends $e(\sigma,\tau)$.

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[20] Cf. the two papers mentioned in footnotes 16 and 17.

[21] E. Witt, "Schiefkörper über diskret bewerteten Körpern," *Crelle*, vol. 176 (1937).

We have $\rho = \nu$ for D_{q^ν} was supposed to be a normal division algebra of degree q^ν, thus the existence of cyclic unramified fields of degree q^ν is established. Moreover we observe that all algebras D_{q^ν} are cyclic and ramified.

THEOREM 7. *If* $[k^*/NZ^*_n:1] = n$ *holds for all cyclic extensions* Z_n *over* k *and similarly for all cyclic extensions over the finite algebraic extensions of* k *then*

(i) *all cyclic unramified fields of degree* n *over* k *coincide,*
(ii) *the field of residue classes* $\boldsymbol{k}$ *is algebraically perfect, and*
(iii) $k^*/NA^* \cong \Gamma(A, k)$ *holds for all abelian extensions* A *over* k.

Proof. First we observe that Lemma 5 implies the existence of cyclic unramified fields Z_q of prime degree q. If $q \neq \chi(\boldsymbol{k})$ then there exist cyclic unramified extensions Z_{q^ν} for arbitrary ν. Namely, either cyclotomic subfields of $k(\zeta_{q^{N+\nu}})$—if ζ_{q^N} is the maximal q^N-th root of unity lying in k—represent such fields or radical extensions $k(a^{1/q^\nu})$ where $Z_q = k(a^{1/q})$ is one of the cyclic unramified extensions which must exist according to Lemma 5. If $q = \chi(\boldsymbol{k})$ then cyclic unramified extensions must exist according to the theory of q-algebras if the latter is applied to perfect fields for which our assumption holds.

Lemma 5 implies furthermore that $\boldsymbol{k}$ is algebraically perfect. Hence there can be established a one to one correspondence between the unramified division algebras over k and the division algebras over $\boldsymbol{k}$.

In order to prove that all cyclic unramified fields of degree n coincide it is sufficient to prove that all cyclic unramified fields of prime degree q^ν coincide for each q. Let then Z_q and Z'_q be two cyclic unramified fields of degree q over k. Their respective groups of algebras $G(Z_q)$ and $G(Z'_q)$ have the generators (π, Z_q) and (π, Z'_q) respectively. Consequently—if $q \neq \chi(k)$—the separable field $k(\pi^{1/q})$ is a common splitting field of $G(Z_q)$ and $G(Z'_q)$. Now assume that k contains the q-th roots of unity then $k(\pi^{1/q})$ is a cyclic extension of degree q over k. Our assumption implies

$$G(Z_q) = G(Z'_q) = G(k(\pi^{1/q})) \cong \mathrm{Z}(q).$$

Since each proper division algebra D_q in $G(k(\pi^{1/q}))$ is ramified and since Z_q and Z'_q are both unramified cyclic splitting fields of D_q we get for the algebra of residue classes

$$\mathfrak{O}/\mathfrak{P} = \mathfrak{O}(Z_q)/\mathfrak{P} = \mathfrak{O}(Z'_q)/\mathfrak{P}, \text{ or } \mathbf{Z}_q = \mathbf{Z}_q' \text{ over } \boldsymbol{k},$$

hence Z_q and Z'_q are abstractly isomorphic which amounts exactly to our assertion.

If k does not contain the q-th roots of unity then consider the extensions of $G(Z_q)$ and $G(Z'_q)$ by the cyclotomic field $k(\zeta_q)$. Since $[k(\zeta_q):k]$ is relatively prime to q the general algebraic theory of splitting fields yields that the algebras in $G(Z_q)$ and $G(Z'_q)$ to not become similar to $k(\zeta_q)$ if the center is extended to $k(\zeta_q)$. The field $k(\pi^{1/q})k(\zeta_q)$ is a cyclic extension of degree q over the new ground field $k(\zeta_q)$. As in the previous case our assumption yields that the extended groups coincide. Hence the fields $Z_q k(\zeta_q)$ and $Z'_q k(\zeta_q)$ which are cyclic of degree $q(q-1)$ over k coincide, consequently according to the Galois theory $Z_q = Z'_q$.

A treatment of the case $q = \chi(\boldsymbol{k}) = \chi(k)$ can be found in the literature.[22]

Next we show that the field of residue classes $\boldsymbol{k}$ is quasi-algebraically closed. Our assumption implies in particular that $k^*/NZ^*_n \cong \mathrm{Z}(n)$ for all cyclic unramified extensions of degree n. Consequently there do not exist cyclic algebras over $\boldsymbol{k}$ for such algebras would be residue algebras of unramified cyclic algebras over k. Furthermore our general assumption implies the same for all finite algebraic extensions over $\boldsymbol{k}$. Hence we can apply Lemma 2 and we see that $\boldsymbol{k}$ is quasi-algebraically closed.

An immediate consequence according to Theorem 1 and the algebraic theory is the fact that all normal division algebras of degree n over k are ramified and that they possess cyclic unramified splitting fields Z_n, moreover degree and exponent of all division algebras coincide. Since the fields Z_n are uniquely determined by their degrees n we see that all algebras which are representable by division algebras of degree n form a cyclic group of order n.

Next we prove that all abelian extensions A_n of degree n over k are splitting fields of the class group of all algebras of degree n over k. Let Z_f be the inertial field of a fixed abelian field A_n, then $\mathfrak{p} = \mathfrak{P}^{n/f}$ in A_n and $\pi = \mathrm{E}\Pi^{n/f}$ with a suitable unit E of A_n. Consider now a generating algebra (π, Z_n, σ_n) of $G(Z_n)$ and form the direct product $(\pi, Z_n, \sigma_n) \times A_n$ then

$$\begin{aligned}(\pi, Z_n, \sigma_n) \times A_n &\sim (\Pi^{n/f}, Z_nA_n/A_n, \sigma_{n/f})(\mathrm{E}, Z_nA_n/A_n, \sigma_{n/f})\\ &\sim (\Pi, Z_nA_n/A_n, \sigma_{n/f})^{n/f} \sim 1\end{aligned}$$

for our assumptions imply that $A^*_n/N(Z_nA_n)^* \cong \mathrm{Z}(n/f)$ and Z_nA_n is the unramified cyclic extension of degree n/f over A_n. Hence A_n is a splitting field of the uniquely determined class group of algebras $G(Z_n)$.

Now we prove that $[k^*/NS^*:1] \leqq [S:k]$ for any solvable extension S of k. Since S is solvable it possesses a chain of subfields

$$k = A_0 \leq A_1 \leq \cdots \leq A_{i-1} \leq A_i \leq \cdots \leq A_{s-1} \leq A_s = S$$

[22] See footnote 20; consider the algebras (a, π^{-1}) which have the same unramified splitting field.

such that always A_i is a cyclic extension over A_{i-1}. The cyclicity of $G(Z_n)$ yields that the norm factor group for any cyclic extension over k is cyclic, according to our general assumption the same is true for all finite algebraic extensions K over k. Thus $[A^*_{i-1}:NA^*_i] = [A_i:A_{i-1}]$. Using the transitive property of the norm and induction by i one readily observes that $[k^*:NS^*] \leqq [S:k]$.[23]

Now let A_n be an Abelian field over k, there always exists a division algebra $D_n = (a(\sigma,\tau), A_n, \Gamma(A_n,k))$ in $G(Z_n)$ as we have seen before. Hence the function $f(\sigma) = \prod_{(\tau)} a(\sigma,\tau)$ determines an isomorphism between $\Gamma(A_n,k)$ and a subgroup of k^*/NA^*_n, consequently

$$k^*/NA^*_n \cong \Gamma(A_n,k)$$

as asserted.[24]

Remark. Our assumptions do not exclude the existence of non-solvable equations in k as the example on page 86 shows.

The uniqueness of the cyclic unramified extensions in Theorem 6 was a consequence of the postulated isomorphism $k^*/NA^* \cong \Gamma(A,k)$. This isomorphism amounts to the fact that the norm factor group of a cyclic unramified extension over another cyclic unramified extension over the ground field k is a cyclic group whose order is equal to the respective relative degree. The same result as in Theorem 6 can be obtained if we postulate for k

(i) $k^*/NZ^*_n \cong Z(n)$ for all cyclic unramified fields,

(ii) if Z_n and Z_m are two cyclic fields then

$$NZ^*_n \cap NZ^*_m = N(Z_nZ_m)^*, \text{ or}$$

(ii′) $\{NZ^*_n, NZ^*_m\} = N(Z_n \cap Z_m)^*$.

Obviously it suffices to prove the uniqueness for cyclic unramified fields Z_{q^ν}' and Z_{q^μ}'' whose degrees are powers of one and the same prime q. Assume that Z_{q^ν}' and Z_{q^μ}'' are distinct. Then

$$q^M < [Z_{q^\nu}'Z_{q^\mu}'':k] \leq q^{\mu+\nu} \quad\text{and}\quad q^m > [Z_{q^\nu}' \cap Z_{q^\mu}'':k] \geq 1.$$

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[23] For a model of the proof see F. K. Schmidt, "Zur Klassenkörpertheorie im Kleinen," *Crelle*, vol. 162 (1930).

[24] T. Nakayama, "Über die Beziehungen zwischen den Faktorensystemen und der Normenklassengruppe eines galoisschen Erweiterungskörpers," *Mathematische Annalen*, vol. 112 (1935).

Since both fields are unramified the group of units ϵ in k is contained in $NZ_{q^{\nu}}'^{*}$ as well as in $NZ_{q^{\mu}}''^{*}$ assumptions (ii) and (ii′) together with (i) yield

$$N(Z_{q^{\nu}}'Z_{q^{\mu}}'')^{*} = q^{M} \quad \text{and} \quad N(Z_{q^{\nu}}' \cap Z_{q^{\mu}}'')^{*} = q^{m},$$

$M = \max(\mu, \nu)$ and $m = \min(\mu, \nu)$. But surely $[k^{*}/N(Z_{q^{\mu}}'Z_{q^{\nu}}'')^{*}:1] > q^{M}$ and $[k^{*}/N(Z_{q^{\mu}}' \cap Z_{q^{\nu}}'')^{*}:1] < q^{m}$. Hence $Z_{q^{m}}$ must be a subfield of $Z_{q^{M}}$ whereby the uniqueness is established.

Lemma 6. *If for each subgroup H of the multiplicative group k^{*} belonging to the perfect field k there exists a uniquely determined abelian field K of finite degree over k such that $NK^{*} = H(K) = H$ and if $H_1 < H_2$ implies $K_2 < K_1$ then*

$$[k^{*}/H:1] = [K:k].$$

Proof. Let H be an arbitrary subgroup of k^{*} and let K be the associated abelian field over k such that $H(K) = H$. The Galois group of K with respect to k be denoted by $\Gamma(K, k) = \Gamma$. Consider now an arbitrary chain of subfields between K and k

$$k = K_0 < K_1 < K_2 < \cdots < K_{i-1} < K_i < \cdots < K_{s-1} < K_s = K$$

such that K_i is a cyclic extension K_{i-1} whose degree is a prime, $i = 1, 2, \cdots, s$. The associated norm groups $N_iK^{*}_i = H(K_i) = H_i$ form a descending chain

$$k = H_0 \geq H_1 \geq \cdots \geq H_{i-1} \geq H_i \geq \cdots \geq H_{s-1} \geq H_s = H$$

as a simple consideration shows. According to our assumption there holds a one to one correspondence between the subgroups H_i and the subfields K_i, K_i being the uniquely determined abelian extension of k belonging to $H_i = N_iK^{*}_i$.

Always $H_{i-1} > H_i$, for if $H_{i-1} = H_i$ then also $K_{i-1} = K_i$ according to the second part of our assumption. Moreover, the factor groups H_{i-1}/H_i are cyclic groups of prime order for the existence of a proper subgroup H' of H_{i-1} which is different from H_i would imply the existence of a field K' such that $K_{i-1} < K' < K_i$.

Hence $[k^{*}/H:1] = \prod_{i=1}^{s} [H_{i-1}/H_i:1]$ is finite.

Now let H be a subgroup of k^{*} such that $[k^{*}/H:1] = q^{\rho}$ where q is a rational prime. We draw a composition series

$$k = H_0 > H_1 > H_2 > \cdots > H_{i-1} > H_i > \cdots H_{\rho-1} > H_{\rho} = H$$

between k^* and H such that the factor groups H_{i-1}/H_i are cyclic groups of order q. The associated abelian extensions of k be

$$k = K_0 < K_1 < K_2 < \cdots < K_{i-1} < K_i < \cdots < K_{\rho-1} < K_\rho = K,$$

evidentally K_i is always a cyclic extension of prime degree q_i over K_{i-1}, $i = 1, 2, \cdots$. Next we show that $q_i = q$. Consider the field K_1 over k then $k > H_1 > k^{*q_1}$, since $[k^* : H_1] = q$ we get $q_1 = q$. Then induction implies $q_2 = \cdots = q_\rho = q$. Hence obviously

$$[k^*/H : 1] = [K : k].$$

Now let H be a subgroup of index $n = \prod_{i=1}^{r} q_i^{\rho_i}$ under k. According to a group theoretical theorem we obtain $H = H^{(1)} \cap \cdots \cap H^{(r)}$ where $H^{(i)}$ are subgroups of index $q_i^{\rho_i}$ under k^*. The Galois theory combined with the assumptions of the Lemma yields that the field K belonging to H is the join of the fields $K^{(i)}$ belonging to the groups $H^{(i)}$. Hence $[k^*/H : 1] = [K : k]$.

Corollary. *For each finite abelian extension K over k holds*

$$[k^*/H(K) : 1] = [K : k].$$

Proof. We draw again a composition chain between k and $H(K)$. The finiteness of the degree $[K : k]$ implies according to the assumptions of the lemma that the index $[k^*/H(K) : 1]$ is finite. The uniqueness of K as the field belonging to $H(K)$ asserts together with Lemma 6 that

$$[k^*/H(K) : 1] = [K : k].$$

Theorem 8. *Under the same assumptions as in Lemma* 6 *it follows that all abelian unramified fields U_n are cyclic.*

Proof. Let U_n be an arbitrary abelian unramified field of degree n over k. Then $H(U_n) = (\{\pi\}^n, N\mathrm{E})$. Since $[k^* : H(U_n)] = n$ according to the corollary we must have $N\mathrm{E} = \epsilon$. The last equality holds for all abelian unramified extensions U_n, hence their norm groups coincide and they must be cyclic for this conclusion holds in particular for all prime degrees q.

Theorem 9. *If the abelian field K over k belonging to the group H is contained in all extensions L over k whose norm groups $H(L)$ are subgroups*

of H and if the assumptions of Lemma 6 hold for k then all unramified finite extensions U_n of k are cyclic.

Proof. Let U_n be an unramified separable field of degree n over k then $H(U_n) = (\{\pi\}^n, N\mathrm{E})$. The group $N\mathrm{E}$ is a subgroup of ϵ the group of all units in k, hence $H(U_n) \subseteq (\{\pi\}^n, \epsilon)$. Let the cyclic unramified extension belonging to $(\{\pi\}^n, \epsilon)$ according to Theorem 8 be Z_n. Then our assumption yields that $Z_n \subseteq U_n$. Consequently $Z_n = U_n$ for both fields have the same degree.

Remark. A further consequence of the assumptions in Lemma 6 is the fact that the field of residue classes $\boldsymbol{k}$ is algebraically perfect. Namely Theorem 8 yields that $k^*/NZ^*_n \cong \mathrm{Z}(n)$ holds for all unramified extensions, hence there do not exist proper division algebras of degree $\chi(\boldsymbol{k})^\nu$ over $\boldsymbol{k}$ as previous considerations show.

The most important conclusion obtained in the previous theorems was the fact that for each integer n there exists a uniquely determined cyclic unramified extension Z_n of degree n over k. In some theorems we had to assume that the field of residue classes be an algebraically perfect field mostly then if we made use of the algebraic theory of division algebras D over k in the proofs. However, in theorems of this type it suffices in general that the assumptions are postulated for all normal division algebras which possess residue division algebras whose centers are separable over $\boldsymbol{k}$. Our aim is to infer from the postulates that $\boldsymbol{k}$ is a Galois field. In order to achieve this it is necessary to impose further conditions on the perfect field k. Namely, there exist examples of perfect fields for which most of the theorems hold although the residue fields are not Galois fields. For example the fields of formal power series in two variables over an algebraically closed field of characteristic 0 where the residue field belonging to only existing isolated subgroup of rank one contained in the valuation group of the field is considered as the residue field of the discrete archimedian valuation of the field.[25]

The restriction to be imposed on k should not contain a postulate with regard to the characteristic of $\boldsymbol{k}$. Thus we are lead to the following assumption *the postulates in the theorems asserting the uniqueness of the unramified cyclic extensions over k shall also be true for all perfect subfields k' of k whose residue class fields $\boldsymbol{k}'$ are subfields of $\boldsymbol{k}$.*

THEOREM 10. *If a field k satisfying any one of the assumptions implying*

[25] O. F. G. Schilling, "Arithmetic in fields of formal power series," *Annals of Mathematics*, vol. 38 (1937).

the uniqueness of the cyclic unramified extensions fulfills the postulate about its perfect subfields then k is a perfect field whose field of residue classes $\boldsymbol{k}$ is a Galois field whose G-number possesses no infinite part.[26]

Proof. The field of residue classes $\boldsymbol{k}$ must have the property that all its subfields $\boldsymbol{k}'$ admit exactly one cyclic extension of degree n, hence in particular the prime field of characteristic $\chi(\boldsymbol{k})$ which lies in $\boldsymbol{k}$. Thus fields $\boldsymbol{k}$ of characteristic 0 are ruled out for the field of all rational numbers possesses infinitely many cyclic extensions for each degree n. For the same reason it follows that $\boldsymbol{k}$ must be algebraic over its prime field if $\chi(\boldsymbol{k}) \neq 0$. Namely, if $\boldsymbol{k}$ would contain a transcendental quantity t over the prime field then the field of all rational functions of t over the prime field had to possess exactly one cyclic extension of degree n for each integer n, an implication which obviously is false.

Now it remains to be proved that the infinite part of the G-number belonging to $\boldsymbol{k}$ is equal to one. Assume that q is a divisor of $G_{\mathrm{inf}}(\boldsymbol{k})$. Then $\boldsymbol{k}$ possesses no cyclic extensions of degree q^{ν} over $\boldsymbol{k}$, consequently the assumptions which lead to the uniqueness of the cyclic unramified extensions over k are violated for all degrees which are divisible by q. An extensive investigation leading up to the preceding assertion has already been made.[27] Hence $G_{\mathrm{inf}}(\boldsymbol{k}) = 1$.

We see that the residue fields $\boldsymbol{k}$ which result are algebraically perfect. Therefore the structure of the perfect fields k is determined by the fields $\boldsymbol{k}$.[28] In a certain sense they are the $\mathfrak{p}$-adic fields belonging to algebraic number fields and fields of functions of one variable whose fields of coefficients are Galois fields; these are exactly the fields for which the local class field theory has first been developed. We wish to mention that the perfect fields which can be characterized by topological properties belong to the class of perfect fields we studied here.[29] Thus we have studied an algebraico-arithmetical counterpart of N. Jacobson's theory.

It was quite material in our investigations that the value group $\mathfrak{V}(k)$ belonging to the perfect field was isomorphic with the additive group of all rational integers. It implies in particular that the prime ideal $\mathfrak{p}$ of k can be generated by a prime element π. Consequently we could speak of a ramifica-

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[26] For the pertaining notations and definitions see the paper mentioned in footnote 1.

[27] See paper mentioned under [1] and an additional note in the same journal.

[28] H. Hasse and F. K. Schmidt, "Die Struktur diskret bewerteter Körper," *Crelle*, vol. 170 (1933).

[29] N. Jacobson, "Totally disconnected locally compact rings," *American Journal of Mathematics*, vol. 58 (1936).

tion degree e belonging to a finite algebraic extension of k or to a normal division algebra over k. However, one can find a substitute for this definition of ramification. Namely, if $e > 1$ then $[\mathfrak{O}/\mathfrak{P} : \mathfrak{o}/\mathfrak{p}] = n/e < n$ in the case that $\mathfrak{B}(k)$ is isomorphic to the additive group of all integers, if $\mathfrak{B}(k)$ is not discrete then we shall say that K is ramified over k if $[\mathfrak{O}/\mathfrak{P} : \mathfrak{o}/\mathfrak{p}] < n = [K : k]$. One easily observes that this is a workable definition for arithmetical problems.

More important are the consequences of the non-discreteness of $\mathfrak{B}(k)$ for the algebraic theory of normal division algebras. We made ample use of the fact that each division algebra which possesses a residue algebra whose center is separable over $\mathbf{k}$ can be represented as the product of a ramified and an unramified algebra

$$\begin{aligned} D &\sim (a(\sigma,\tau), U, \Gamma(U,k)) \\ &\sim (\pi^{e(\sigma,\tau)}, U, \Gamma(U,k))(\eta(\sigma,\tau), U, \Gamma(U,k)). \end{aligned}$$

For general value groups $\mathfrak{B}(k)$ of rank one such a decomposition is not obvious for there do not exist generating prime elements of the prime ideal belonging to the valuation $\mathfrak{B}(k)$. Nevertheless, under restricting conditions on the structure of the field k we are able to find a substitute for such decompositions; but it turns out that such decompositions can no more be made unique as it was possible in the case of discrete valuations by fixing the prime element π. Let us assume that there exists at least one discrete perfect subfield k' of k which has the same field of residue classes $\mathbf{k}$ as the given field k, and that there exists a—necessarily infinite—algebraic extension k'' of k' which is everywhere dense in k, that is to say whose derived field with respect to the valuation is equal to k. The values assumed by the elements of k'' form a group of rational numbers $\mathfrak{B}_e(k'')$ whose structure is determined by the G-degree of k'' over k'. In order to see that one has to observe that any number of k'' lies in a finite perfect extension of k' and that its value is determined by the relative degree of that extension and that the value does not depend upon the particular finite field used for its determination. Analyzing the structure of the group $\mathfrak{B}_e(k'')$ we see that certain elements of it may possess arbitrarily high powers of primes as denominators. The set of such rational primes shall be called the infinite characteristic of $\mathfrak{B}_e(k'')$, it is determined by the infinite part of the G-degree $[k'' : k']$; we collect all these primes q in a formal product $\prod_{(q)} q = G$.

Now it can be shown that each algebra D over k whose algebra of residue classes has a separable center over $\mathbf{k}$ is equivalent to the direct product of a

suitable algebra D'' over k'' with k.[30] Thus the study of normal algebras over k is reduced to the investigation of the normal algebras over k''.

Now let D'' be an arbitrary division algebra over k'',

$$\begin{aligned}&D'' \sim (a(\sigma,\tau)'', U'', \Gamma(U'', k'')),\\ &U'' = k''(A'') \text{ where } A''^{n} + a''_1 A''^{n-1} + \cdots + a''_n = 0 \text{ with } a''_i \text{ in } k''.\end{aligned}$$

Consider now an arbitrary finite algebraic extension k_* of k' which contains the elements $a(\sigma,\tau)''$ and a''_i. Then $U'' = k_*(A'')k''$ and

$$D'' \sim (a(\sigma,\tau)'', k_*(A'')/k_*, \Gamma) \times k'' \sim D_* \times k''.$$

Since k_* is a finite algebraic extension of the discrete perfect field k' there exists a prime element π_* in k_*, hence

$$D_* \sim (\pi_*^{e(\sigma,\tau)}, k_*(A'')/k_*, \Gamma)\,(\eta(\sigma,\tau)_*, k_*(A'')/k_*, \Gamma),$$

where the first factor represents a ramified algebra and the second factor stands for an unramified algebra. The algebras

$$\begin{aligned}&(\pi_*^{e(\sigma,\tau)}, k_*(A'')/k_*, \Gamma) \times k'' \quad \text{and}\\ &(\eta(\sigma,\tau)_*, k_*(A'')/k_*, \Gamma) \times k''\end{aligned}$$

are at most similar to a ramified or an unramified division algebra over k'', respectively, where the ramification is understood to be measured according to

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[30] Cf. Note 1. The result essentially used here can be generalized as follows: "If k'' is an everywhere dense subfield of k such that the residue class fields of k'' and k coincide, every separable extension $k(A)$ of degree n over k is equal to the join of k with a suitable separable extension $k''(A'')$ of degree n over k''." A proof of this theorem can be obtained by generalizing a proof of M. Moriya in "Klassenkörpertheorie im Kleinen fur die unendlichen algebraischen Zahlkörper" (*Journal of the Fac. of Sci. Hokkaido Imp. Univer.*, ser. I, vol. 5 (1936), Sapporo, Japan). Using the notations *loc. cit.*, p. 13, we observe that one can take instead of k any everywhere dense subfield k'' of k. Furthermore, in the general case one has to use for the construction of the equation for A'' the theory of abstract derivation as developed by H. Hasse in "Noch eine Begründung der Theorie der höheren Differentialquotienten in einem algebraischen Funktionenkörper einen Unbestimmten" (*Crelle*, vol. 177 (1937). The proof that the constructed equation for A'' is irreducible in k'' resp. k can be reduced to a theorem of F. K. Schmidt stating that two polynomials of degree n whose distance is sufficiently small in the metric space of all polynomials of degree n, have the same type of decomposition. Cf. F. K. Schmidt, "Mehrfach perfekte Körper" in *Mathematische Annalen*, vol. 108 (1933). Our theorem is rather helpful for the investigation of the structural theory of general perfect fields whose value groups are isomorphic with non-discrete subgroups the additive group of all real numbers.

the new definition. The resulting decomposition of D'' depends essentially upon the choice of the subfield k_*, nevertheless the distinction between the ramified and unramified part still persists.

$$(\pi_*^{e(\sigma,\tau)}, k_*(A'')/k_*, \Gamma) \times k'' \sim (\pi_*^{\nu}, Z_*/k_*, \sigma) \times k'',$$

where Z_* denotes the cyclic unramified subfield of $k_*(A'')$ belonging to the character induced by $e(\sigma, \tau)$. The cyclic field $Z'' = Z_* k''$ can also be described as the subfield of U'' which is determined by the abelian character of Γ singled out by the values of the original factor set $a(\sigma, \tau)''$ within $\mathfrak{B}(k)$ or $\mathfrak{B}(k'')$ for the exponents $e(\sigma, \tau)$ are uniquely determined modulo $[U'' : k]$ within $\mathfrak{B}(k_*)$ or $\mathfrak{B}(k'')$.

If the prime r does not divide G, then we can select the field k_* used for the construction of a division algebra D'' whose degree is a power of r such that the relative degrees of all fields between k_* and k'' are relatively prime to r. Hence k'' does not split the related algebra $D_* \not\sim k_*$, of course provided that such an algebra exists.[31] In such a case the algebra D'' is represented as the direct product of a ramified and an unramified algebra as we observed before.

Now let q be a divisor of G, and let $D_* = (\pi_*, Z_{q^\nu}, \sigma_{q^\nu})$ be a division algebra over k_*. Here we can no more assume that the relative degrees of all fields between k_* and k'' are relatively prime to q. Let K be a sufficiently large extension of k_* such that $[K : k_*] = q^\lambda s$ is divisible by q^ν. Then

$$D_* \times K \sim (\Pi^{q^\lambda s} \mathrm{E}, Z_{q^\nu} K/K, \sigma_{q^\nu}) \sim (\mathrm{E}, Z_{q^\nu} K/K, \sigma_{q^\rho}),$$

where Π denotes a prime element of the field K and where $[Z_{q^\nu} K : K] = q^\rho$. Hence the algebra $D_* \times K$ and a fortiori the algebra $D_* \times k''$ are equivalent to unramified algebras over K and k'' respectively. Thus we see that there do not exist ramified division algebras D'' over k'' whose degree is a power of a prime q dividing the infinite characteristic G. Observe that there may very well exist unramified division algebras over k'' whose degree is a power of such a prime q.

We next wish to construct an example of a perfect field k whose value group $\mathfrak{B}(k)$ is not discrete and where all normal algebras of degree n form a cyclic group of order n. Let $\boldsymbol{k}$ be a finite p-adic number field, then all classes of algebras which can be represented by normal division algebras whose degrees are divisors of n form a cyclic group of order n for each integer n. Consider

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[31] For a more detailed treatment of the perfect fields arising from infinite algebraic number fields see [1].

the field $k' = \boldsymbol{k}\{t\}$ of all formal power series in t which possess coefficients in the field $\boldsymbol{k}$. The adjoin to k' all radicals t^{1/q^ν} where q runs over all rational primes and $\nu = 1, 2, \cdots$. The resulting enumerable infinite extension k'' possesses an infinite characteristic G which is divisible by all rational primes. According to what we have seen before all division algebras of degree n are unramified and consequently they are determined by the division algebras $\boldsymbol{D}$ over $\boldsymbol{k}$, since $\boldsymbol{k}$ is algebraically perfect there can be established a one to one correspondence between the algebras over k'' and $\boldsymbol{k}$. Thus we see that all classes of algebras over k'' which can be represented by division algebras whose degrees are divisors of n for a cyclic group of order n. The same is true for the classes of normal algebras over the derived field k belonging to k''. One readily observes that the local class field theory in the form of the isomorphism theorem stating the isomorphism of the Galois group of an abelian extension A over k with the respective norm class group in k does not hold. Thus we observe that the discreteness of the value group $\mathfrak{B}(k)$ is essential for the validity of all theorems known in the usual local class field theory.

As a matter of fact one readily can construct examples of fields k which are perfect with respect to a non-discrete valuation of rank one—or of fields which are everywhere dense in such fields—such that certain algebras are ramified and others are unramified according to the respective degrees. One has to consider suitable infinite perfect fields $\boldsymbol{k}$ for which the local class field theory holds in part, i. e. such that all algebras whose degrees divide a fixed G-number form cyclic groups, then one adjoins to $\boldsymbol{k}\{t\}$ sufficiently many algebraic quantities such that the infinite characteristic of the resulting field k'' is equal to the previously fixed G-number and such that the algebras over k'' whose degrees are relatively prime to the G-number form cyclic groups. The resulting field k'' has the property that all algebras whose degrees divide a fixed integer n form a cyclic group of order n. Moreover, examples of such a type show that the cyclicity of class groups of algebras over a field does not imply that the field is perfect. Repeating the process of adjoining transcendental quantities like t one can find fields which admit a non-discrete valuation of arbitrary type of ordering but such that the special property of the class groups of algebras holds.

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GROUPS WHOSE COMMUTATOR SUBGROUPS ARE OF ORDER TWO.*

By G. A. Miller.

If the commutator subgroup of a group G is of order two the commutators of G are invariant and hence every operator of odd order appears in the central of G since such an operator could not be transformed into an operator of even order. It therefore results that when G involves operators of odd order it is the direct product of a group of order 2^m and of an abelian group of odd order. It is desirable to exclude direct products in what follows since all except one of the factor groups would be abelian. Hence we shall assume hereafter that the order of G is 2^m and that G is not a direct product. The central of G includes the squares of all the operators of G and hence the central quotient group of G is abelian and of type 1^n, where n is even since the subgroup composed of all of the operators of G which are commutative with two of its non-commutative operators is of index 4 under G.

It is possible to construct as follows a G whose central is an arbitrary abelian group. If $t_1, t_2, \cdots, t_l$ is a set of restricted independent generators of this abelian group we divide these operators into distinct pairs when l is even or we divide $l-1$ of them into distinct pairs when l is odd. In the former case we construct two operators whose squares are the operators of such a pair and that one of these two operators generates the commutator subgroup. Each of these two operators may be assumed to transform the other into itself multiplied by the commutator of order 2. The remaining pairs of independent generators may be assumed to be such that none of them generates the commutator subgroup but that each of the operators of a pair is the square of an operator of G and that two such operators are again non-commutative but are commutative with all of the other operators thus constructed.

We thus arrive at a G which has the given abelian group for its central and is not a direct product of two groups since its central is generated by the squares of its operators. When l is odd we may proceed similarly with the exception that the operator which does not appear in a pair may be assumed to be the only one of the set of independent generators which separately

* Received Aug. 3, 1937.

generates the commutator subgroup and is a non-square. This group is again not a direct product since if it were a direct product one of the factor groups would be abelian and would not involve the commutator of order 2. Hence there results the following theorem: *It is possible to construct a group of order 2^m which has an arbitrary abelian group whose order is of this form as its central and has a commutator subgroup of order 2 but is not the direct product of two groups.* Whenever two G's have centrals which are not simply isomorphic the G's are not simply isomorphic but there are also non-simply isomorphic G's which have simply isomorphic centrals.

The G's which have for their common central the group of order 2 are identical with the non-abelian groups which separately satisfy the condition that the squares of their operators constitute the group of order 2. It is known that there are three infinite systems of such groups.[1] We proceed to determine the G's which have for their centrals the group of type 1^n, where $n > 1$, and shall first impose the additional condition that each of the operators of order 2 contained in such a group is invariant. Let s_1 and s_2 represent two non-invariant operators of order 4. The group generated by s_1 and s_2 may be of order 8, 16, or 32. When it is of order 8 it is the quaternion group. When it is of order 16 it contains 12 operators of order 4 which have two distinct squares and hence it is completely determined. When it is of order 32 the squares of s_1 and s_2 do not generate the commutator subgroup and hence it is again completely determined.

Suppose now that G contains the quarternion group. Its subgroup composed of its operators which are commutative with all the operators of this quaternion group cannot involve any operator whose square is the commutator of order 2. If this subgroup involves two non-commutative operators of order 4 they therefore generate the given group of order 32. This is also the case when the subgroup of index 4 under this subgroup composed of all its operators which are commutative with each of these two non-commutative operators involves two non-commutative operators of order 4, etc. By continuing this process we finally arrive, when G is of finite order, at an abelian subgroup of type 1^n, n being of the form $2m + 1$, where $m + 1$ represents the number of these successive steps. Such a system can readily be constructed by starting with the abelian group of type 1^{2m+1} and inserting the quaternion group at an arbitrary stage of the process. The order of G is 2^{4m+3}, where m is an arbitrary positive integer or zero. This system of groups is characterized by the following properties. Each group has the group of order 2 for its commutator sub-

[1] G. A. Miller, *American Journal of Mathematics*, vol. 55, pp. 417-420.

group, involves the quaternion group, contains only operators of order 2 and 4 besides the identity, every operator of order 2 is in the central and every operator of order 4 is non-invariant, there is one and only one group for a given positive or zero value of m.

When G does not involve two non-commutative operators of order 4 which generate the quaternion group but contains two such operators which generate the given group of order 16 then we can proceed similarly to construct an infinite system of groups which satisfy the conditions in question. In this case we arrive at the abelian group of type $1^{2(m+1)}$, and each of the successive groups, after the first, is again of order 32. This system is characterized just as the preceding one except that the quarternion group therein is replaced by the given group of order 16 and the order of G is $2^{4(m+1)}$. Finally, when every two non-invariant operators of G generate the given group of order 32 there results the third and last system of groups which are separately characterized by the facts that each group has the group of order 2 for its commutator subgroup, contains no operator whose order exceeds 4, all of its operators of order 2 are in its central while each of its operators of order 4 is non-invariant. The order of each of these groups is 2^{4m+5}, where m is a positive integer or zero, and there is one and only one such group for every such value of m. That is, *there are three and only three infinite systems of groups which separately satisfy the three conditions that each of them has the group of order 2 for its commutator subgroup, contains no operator whose order exceeds 4, all its operators of order 2 are in its central but none of its operators of order 4 appears therein.*

We proceed to consider the groups which are characterized by the conditions that their centrals are of type 1^n, $n > 1$, and that they separately involve non-invariant operators of order 2. Such a G is generated by its operators of order 4 since these operators could not generate a proper subgroup such that each of the remaining operators is of order 2 since this proper subgroup would be abelian and its operators would be transformed into their inverses by the remaining operators of G. When all the operators of order 2 contained in G are relatively commutative they generate an invariant abelian subgroup of G. As G is supposed to contain operators of order 2 which are not commutative with all of its operators of order 4 it results that it contains two non-commutative operators of order 4 which generate a group of order 16 involving non-invariant operators of order 2. Hence the following theorem: *If the central of a group whose commutator subgroup is of order 2 is of type 1^n and if this group involves non-invariant operators of order 2 but all of its operators of this order are relatively commutative then it contains invariantly*

the group of order 16 *which is characterized by the fact that it contains exactly eight operators of order* 4 *which have two distinct squares.*

When the operators of order 2 contained in G generate an abelian subgroup then G can be constructed by successively extending this subgroup by operators of order 4 which are commutative with half of the operators of this subgroup and whose squares do not include the commutator of order 2 contained in G. This process of extending this subgroup and the resulting group is continued until we arrive at a group whose central is generated by the squares of the added operators of order 4. The extending operator in each case has a square which does not appear in the group of the squares of the operators of order 4 previously constructed. If the original abelian subgroup is of order 2^k the number of these extensions is $k - 1$ and the central of each extended group is half of the central of the preceding group. The smallest group in this system is the group of order 16 noted at the close of the preceding paragraph, and the central of each of these groups contains one and only one operator of order 2 which is not a square, namely the commutator of this order.

Suppose that a group whose commutator subgroup is of order 2 and whose central is of type 1^n is generated by its operators of order 2. Two non-commutative operators of order 2 contained in such a G generate the octic group. The subgroup of G composed of all its operators which are commutative with every operator of this octic group is also generated by its operators of order 2 since G is not a direct product. Hence there results the theorem that if a group whose commutator subgroup is of order 2 and whose central is of type 1^n is not a direct product it belongs to the infinite system of groups which is characterized by the facts that more than half of the operators of each group are of order 2 and that the squares of the operators of each group constitute the group of order 2. It therefore results that the groups under consideration are not separately generated by their operators of order 2.

The groups whose commutator subgroups are of order 2 and whose centrals are of type 1^n, $n > 1$, therefore have the property that they involve proper subgroups generated by their operators of order 2 whenever they involve non-invariant operators of this order. Such a proper subgroup involves invariant operators of order 2 besides the commutator of this order and the squares of its operators of order 4, and G can be constructed by extending such a subgroup by operators of order 4 which have different squares and whose squares do not include the commutator of order 2. Such an extending operator is non-commutative with an invariant operator of order 2 of this subgroup which is not the commutator nor a square. If the resulting group contains an invariant operator of order 2 which is not a commutator or a square this extension is repeated until the resulting group contains no invariant operator

of order 2 besides the commutator of this order and those which are squares. By these methods all the groups whose centrals are of type 1^n, $n > 1$, and whose commutator subgroups are of order 2 can be constructed.

It was noted above that the squares of all the operators of G appear in the central of G. When the commutator of order 2 in G is the square of an invariant operator of G then the squares of the operators of G generate a subgroup composed of operators which are squares. This results from the fact that if s and t represent any two operators of such a G then s^1t^2 is the square of an operator of G since when s and t are non-commutative then s^2t^2 is the square of st multiplied by an invariant operator of order 4 which generates the commutator of order 2. The central of G may contain operators whose order exceeds 2 which are not squares but generate the commutator of order 2. It may also contain operators which are not squares and do not generate the commutator of order 2. In the latter case such an operator generates also the square of a non-invariant operator of G since G would otherwise be a direct product. Hence there results the following theorem: *If the central of a group whose commutator subgroup is of order 2 contains operators which are not squares of other operators of the group then such an operator either generates the commutator of order 2 or the square of a non-invariant operator whose order exceeds 2.*

If s is a non-invariant operator of lowest order in G then s can not generate the commutator of order 2 if G contains an operator of larger order which generates this commutator. If there is another operator in G of the same order as s which is non-commutative with s then this operator can not generate the commutator of order 2 in G if its order exceeds 4. Hence it results that we can select a set of operators of G in the following manner: The first two are non-commutative operators of lowest order in G, the second two are non-commutative operators of lowest order in the subgroup of index 4 composed of all the operators of G which are commutative with the first two selected operators, etc. Then if one of these operators generates the commutator of order 2 contained in G none of the other operators of the set can have this property unless the two operators are of order 4.

From what precedes it results that the groups whose commutator subgroup is of order 2 can be divided into two categories. In one of these the commutator of order 2 is generated only by operators in the central while in the other this commutator is generated by some non-invariant operator. In the latter case a set of independent generators of each of the groups can be so selected that one and only one of them generates the commutator of order 2 unless two of them appear in the quaternion group. In this case none of the remaining operators of the set generates the commutator of order 2. Hence

the construction of these groups is to a large extent reduced to the construction of groups generated by two non-invariant operators in addition to the central. Therefore we proceed to consider this special case.

If these two non-invariant operators are in the quaternion group then the central may be any cyclic group which generates the commutator of order 2 but it can be no other group. When the order of this cyclic group exceeds 2 the group is also generated by two non-invariant operators which appear in the octic group together with the same central cyclic group. When each of the two non-invariant generating operators has an order which exceeds 4 and one of them generates the commutator of order 2 the central may be an arbitrary abelian group having at most two invariants, which are at least equal respectively to half the orders of these two independent generators. Finally, when neither of the two non-invariant generating operators generates the commutator of order 2 the central may be an arbitrary abelian group having at most three invariants. One of the three independent generators of the central, in the case that there are three such generators, then generates the commutator of order 2 and has an arbitrary order while the orders of the other two are respectively at least equal to one-half of the orders of the two non-invariant generators of G.

University of Illinois.

CONVERGENCE OF A SEQUENCE OF LINEAR TRANSFORMATIONS.*

By M. H. Ingraham and M. C. Wolf.[1]

The purpose of the present paper is to study conditions on a sequence $\{A_i\}$ of $n \times n$ matrices with elements in the complex field, such that if an infinite sequence of linear transformations with matrices $\{A_i\}$ is applied to any bounded region of an n-dimensional vector space, the region converges uniformly to the origin. Certain phases of the problem are discussed in general for n-space and carried out in detail for $n = 2$.

The work on this paper was suggested by a problem in the theory of genetics.[2]

I. Sufficient conditions in the general case. An n-dimensional vector ξ after the application of a linear transformation with $n \times n$ matrix A is the vector $A\xi$ of length $\sqrt{\xi'\bar{A}'A\xi}$ with direction components which are in general different from those of ξ. If $A\xi_i = \lambda_i\xi_i$, then ξ_i is an invariant direction (characteristic direction or vector) of the transformation with matrix A, and is said to correspond to the characteristic value λ_i. In general, for convenience, the vectors will be taken unitary.

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[1] The work of M. C. Wolf was supported by a grant from the Wisconsin Alumni Research Foundation.

[2] For a purely Mendelian case of a characteristic determined by one gene, there are three possible types: 1) homozygous dominant, 2) heterozygous, and 3) homozygous recessive. The proportions of these three types in a population can be represented by a point in a plane, or in the case of n genes by a point in space of $2n$ dimensions.

Under certain hypothesis as to selective mating and productivity the coördinates of the point representing one generation can be given as quotients of quadratic forms in the coördinates of the point representing the preceding generation.

Invariant points represent conditions of equilibrium. These may be of stable or unstable type depending on whether or not the invariant point is the limit of the iterate of the transformation operating on all neighboring points. This may be studied by means of the iterates of the Jacobian of the transformation at the invariant point. As the transformations for any case but that of an " infinite population " vary slightly from generation to generation within very small limits, a better picture of the actual case is obtained by studying the products of a finite number of nearly equal transformations taken from an infinite sequence of such transformations rather than the iterates of a single transformation.

When $\bar{\xi}'_1\xi_1 = \bar{\xi}'_2\xi_2 = 1$, then $|\bar{\xi}'_1\xi_2| \leq 1$. Hence for any matrices R and S and any unitary vector ξ, it follows that for all unitary η,

$$(1) \qquad |\bar{\xi}'RS\xi| \leq \sqrt{\max \bar{\eta}'\bar{R}'R\eta \ \max \bar{\eta}'\bar{S}'S\eta}.$$

If A is an $n \times n$ matrix with characteristic values λ_i $(i = 1, 2, \cdots, n)$, and if the columns of a non-singular matrix T are a set of corresponding characteristic directions $\xi_1, \xi_2, \cdots, \xi_n$, then $A = TBT^{-1}$ where

$$B = \begin{bmatrix} \lambda_1 & & & & 0 \\ & \lambda_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ 0 & & & & \lambda_n \end{bmatrix}.$$

If $\bar{\xi}'\xi = 1$, and if $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$, then $\bar{\xi}'\bar{B}'B\xi \leq N(\lambda_1)$, where $N(\lambda_1)$ is the norm of λ_1. Since $A = TBT^{-1}$ it follows from (1) that

$$(2) \qquad \bar{\xi}'\bar{A}'A\xi \leq N(\lambda_1)K^2$$

where K is a function of the columns of T, and hence depends only on the characteristic directions of A. K may be taken greater than unity. It is important to note that $N(\lambda_1)K^2$ is an upper bound for the Hermitian form $\bar{\xi}'\bar{R}'R\xi$ for unitary ξ associated with any matrix R for which the characteristic directions are given by the above matrix T and for which $N(\lambda_1)$ is the maximum norm of the characteristic values.

THEOREM 1. *The unitary vectors converge uniformly toward the origin under a sequence of transformations with matrices* $\{A_i\}$ *where* $A_i = A + E_i$, *if* $\{E_i\}$ *is a sequence of* $n \times n$ *matrices such that* $\bar{\xi}'\bar{E}'_iE_i\xi \leq H^2$ *for all* i *and all unitary* ξ, $N(\lambda_1)$ *and* K *are the constants described above, and if* H *is positive and satisfies the condition* $[N(\lambda_i) + KH] < 1$.

Let

$$C_k = (A + E_k)(A + E_{k-1}) \cdots (A + E_2)(A + E_1) = \sum_{i=0}^{k} P_i$$

where P_i is a sum of $\binom{k}{i}$ terms; each term of P_i is a product of k matrices in which the matrix A is a factor $k - i$ times and the remaining i factors are i of the matrices $E_1, E_2, \cdots, E_k$. For example, if $k = 9$ and $i = 3$, $A^2E_7AE_5E_4A^3$ is a typical term of P_3.

$$\bar{C}'_k = \sum_{i=0}^{k} \bar{P}'_i$$

$$\bar{\xi}'\bar{C}'_kC_k\xi = \bar{\xi}' \sum_{l=0}^{2k} Q_l\xi = \sum_{l=0}^{2k} \bar{\xi}'Q_l\xi$$

where each Q_l is a sum of $\binom{2k}{l}$ products of the form $X_1X_2\cdots X_kY_1Y_2\cdots Y_k$. Of the X_i, $k-s$ are $\bar{A}'$ and s are certain of the $\bar{E}'$; $k-t$ of the Y_i are A and the remaining t are certain of the E_i; $t+s=l$. To determine a bound for $|\bar{\xi}'Q_l\xi|$ apply (1). Let $N(\lambda_1)=M^2$. The bound for $\sqrt{\bar{\xi}'\bar{A}'^iA^i\xi}$ as determined from (2) is $\sqrt{[N(\lambda_1)]^iK^2}=M^iK$; $\sqrt{\bar{\xi}'\bar{E}'_iE_i\xi}\leqq H$; K enters the bound for $\bar{\xi}'Q_l\xi$ for every separate power A^p or $\bar{A}'^p$ in Q_l, but the factors $\bar{A}'^p$ in the product $X_1X_2\cdots X_k$ are separated by $\bar{E}'_i$'s at most $s+1$ times and the factors A^p in $Y_1Y_2\cdots Y_k$ are separated by E_i's at most $t+1$ times, hence it follows that if $K\geqq 1$

$$|\bar{\xi}'Q_l\xi| < \binom{2k}{l} M^{k-s}K^{s+1}H^s\cdot M^{k-t}K^{t+1}H^t$$

$$|\bar{\xi}'Q_l\xi| < K^2\cdot\binom{2k}{l} M^{2k-l}H^lK^l.$$

For example, suppose a term of Q_5, for $k=8$, is

$$T_5=\bar{A}'^3\bar{E}'_4\bar{E}'_5\bar{A}'\bar{E}'_7\bar{A}'\cdot A^4E_4A^2E_1.$$

$$|\bar{\xi}'T_5\xi| \leqq (M^3K\cdot H\cdot H\cdot MK\cdot H\cdot MK)\cdot(M^4K\cdot H\cdot M^2K\cdot H)=M^{11}K^5H^5$$

$$|\bar{\xi}'T_5\xi| < K^2\cdot M^{11}K^5H^5.$$

Clearly when $K\geqq 1$

$$\bar{\xi}'\bar{C}'_kC_k\xi < K^2(M+HK)^{2k}.$$

If $K<1$, then

$$\bar{\xi}'\bar{C}'_kC_k\xi < (M+HK)^{2k}.$$

If $(M+HK)<1$, $\lim\limits_{k\to\infty}(M+HK)^{2k}\to 0$. Hence if $|\lambda_i|<1$ and if $A+E_i$ differs but slightly from A, that is, if $\bar{\xi}'\bar{E}'_iE_i\xi\leqq H^2$ is sufficiently small, the region about the origin will converge uniformly to the origin under the sequence of transformations with matrices $\{A_i\}$.

In the remainder of this section more detailed conditions for convergence and divergence of vector spaces will be given.

THEOREM 2. *If $\{g_i(A)\}$ is a sequence of polynomials in the above restricted matrix A, and if $\lim\limits_{m\to\infty}N(\rho_m)\to 0$ where $N(\rho_m)$ is the maximum norm of the characteristic values of $\prod\limits_{i=1}^{m}g_i(A)$, then any bounded portion of space*

tends uniformly toward the origin under the sequence of transformations with matrices $\{g_i(A)\}$.

This follows from (2) since the characteristic directions of $\prod_{i=1}^{m} g_i(A)$ are those of A; furthermore, if λ_i is a characteristic value of A, $g_j(\lambda_i)$ is a characteristic value of $g_j(A)$ and $\prod_{j=1}^{m} g_j(\lambda_i)$ is a characteristic value of $\prod_{j=1}^{m} g_j(A)$.

COROLLARY 2. 1. *If A, restricted as above, is a matrix with characteristic values λ_i such that $N(\lambda_i) < 1$, then under iteration of the transformation whose matrix is A, the whole space tends uniformly toward the origin, and if $N(\lambda_i) > 1$, the space diverges.*

THEOREM 3. *If $\{A_i\}$ is a sequence of matrices with characteristic values λ_{ij}, the condition that $|\lambda_{ij}| < 1$, for every i and j, is not sufficient to insure that a bounded region in space will tend to the origin under the transformation with matrices $\{A_i\}$.*

An example of such a sequence may be constructed by iteration of two transformations alternately. The first takes a unitary vector ξ into η where $\bar{\eta}'\eta = k^2 > 1$ and the second rotates η into $l\xi$ where $k^2 > l^2 > 1$.

On the other hand, there exist sequences $\{A_i\}$ of matrices for which the norm of one characteristic value for every A_i is greater than unity, yet the unitary vectors converge uniformly to the origin. The sequence which is alternately A_1 and A_2 is an example of such when

$$A_1 = \begin{pmatrix} 1/3 & 0 \\ 0 & 3/2 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 3/2 & 0 \\ 0 & 1/3 \end{pmatrix}.$$

THEOREM 4. *If $\{A_i\}$ is a sequence of matrices such that*

$$\|A_i\| = |\lambda_{i1}\lambda_{i2}\cdots\lambda_{in}| > 1,$$

then the set of unitary vectors will not converge to the origin under the sequence of transformations with matrices $\{A_i\}$.

Let $B_m = A_m A_{m-1} \cdots A_1$.

$$\|B_m\| = \|A_m|\cdot|A_{m-1}|\cdots|A_1\| > 1.$$

It suffices to prove that there exists at least one vector η such that $\bar{\eta}'\bar{B}'_m B_m \eta > 1$ for infinitely many values of m. Let

$$C_m = \begin{bmatrix} \sigma_1(m) & & & & 0 \\ & \sigma_2(m) & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ 0 & & & & \sigma_n(m) \end{bmatrix} = \bar{T}'_m \bar{B}'_m B_m T_m,$$

$\bar{T}'_m T_m = I$. The set of functional values of $\bar{\zeta}' C_m \zeta$ is equal to those of $\bar{\zeta}' \bar{B}'_m B_m \zeta$ for unitary ζ. Consider real unitary vectors η with coördinates $x_1, x_2, \cdots, x_n$.

Suppose $x_1 \neq 0$ and

$$(3) \qquad \sigma_1(m) \geq \sigma_2(m) \geq \cdots \geq \sigma_n(m) \geq 0.$$
$$| C_m | = \sigma_1(m)\sigma_2(m) \cdots \sigma_n(m) > 1$$

is an increasing function of m. Hence $| C_m | > k > 1$ for some k and $\sigma_1(m) > \sqrt[n]{k} > 1$. From (3), $\bar{\eta}' C_m \eta \geq \sigma_1(m) x_1^2$. If $x_1 > \dfrac{1}{\sqrt[2n]{k}}$, $\bar{\eta}' C_m \eta > 1$. It follows that there exists a vector ζ_m and an ε_k, where ε_k is independent of m, such that $\bar{\zeta}'_m \bar{B}'_m B_m \zeta_m > 1$ and $\bar{\zeta}'_i \bar{B}'_m B_m \zeta_i > 1$ for all ζ_i for which

$$\left| \frac{\bar{\zeta}'_i \zeta_m}{\sqrt{\bar{\zeta}'_i \zeta_i \cdot \bar{\zeta}'_m \zeta_m}} - 1 \right| < \varepsilon_k.$$

The infinite sequence of vectors ζ_m $(m = 1, 2, \cdots)$ has a limit vector ζ. Let ζ_{1i} be an infinite subsequence of ζ_m such that

$$\left| \frac{\bar{\zeta}' \zeta_{1i}}{\sqrt{\bar{\zeta}' \zeta \cdot \bar{\zeta}'_{1i} \zeta_{1i}}} - 1 \right| < \varepsilon_k,$$

then $\bar{\zeta}' \bar{B}'_{1i} B_{1i} \zeta > 1$ for an infinite subsequence $\{B_{1i}\}$ of $\{B_i\}$, as was to be proved.

The proof of the next theorem is similar to that of Theorem 2.

THEOREM 5. *If $\{A_i\}$ is a sequence of matrices such that the characteristic values λ_{ij} $(j = 1, 2, \cdots, n)$ for all i correspond respectively to the linearly independent invariant directions $\xi_1, \xi_2, \cdots, \xi_n$ and if $|\lambda_{ij}| < k < 1$, the unitary vectors converge uniformly to the origin.*

The point of view of this paper is intimately connected with the Cauchy-Riemann conditions of analytic function theory. A 2×2 matrix $A = (a_{ij})$ may be interpreted as the Jacobian matrix of a transformation $U = u(x, y)$ and $V = v(x, y)$. At a point the Cauchy-Riemann conditions are that

$a_{11} = a_{22}$ and $a_{12} = -a_{21}$. If $\begin{pmatrix} 1 \\ m_1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ m_2 \end{pmatrix}$ are the invariant directions of A, m_1 and m_2 satisfy the equation $x^2 + 1 = 0$ unless $a_{12} = 0$, in which case A is of the form λI where λ is real. The Cauchy-Riemann conditions are therefore equivalent to having

$$\xi_1 = \begin{pmatrix} 1 \\ i \end{pmatrix} \quad \text{and} \quad \xi_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

as characteristic vectors and λ_1 and $\lambda_2 = \bar{\lambda}_1$ as characteristic values. It is clear that these conditions give conformal representation, since if

$$\eta_1 = c_{11}\xi_1 + c_{12}\xi_2 \quad \text{and} \quad \eta_2 = c_{21}\xi_1 + c_{22}\xi_2,$$

then

$$\frac{\bar{\eta}'\bar{A}'A\eta_2}{\sqrt{\bar{\eta}'_1\bar{A}'A\eta_1}\,\sqrt{\bar{\eta}'_2\bar{A}'A\eta_2}} = \frac{\bar{\eta}'_1\eta_2}{\sqrt{\bar{\eta}'_1\eta_1}\,\sqrt{\bar{\eta}'_2\eta_2}}$$

which is the condition that the angle between η_1 and η_2 is preserved (except possibly for sign). This equality follows since $\bar{\xi}'_1\xi_2 = \bar{\xi}'_2\xi_1 = 0$, whence also for $i, j = 1, 2$

$$\bar{\eta}'_i\bar{A}'A\eta_j = \lambda_1\bar{\lambda}_1\bar{\eta}'_i\eta_j.$$

II. Characteristic directions of products in real 2-space. It is seen that the conditions under which the whole space under successive transformations tends toward the origin depend not only on the maximum stretch under any one transformation, but also upon the relations of the characteristic directions of the various transformations. It is plausible therefore that if the characteristic vectors of a series of matrices were nearly the same, results could be secured which are analogous to the case in which they are equal. Therefore this section is devoted to the study of the characteristic vectors and values of the product of 2×2 matrices.

Suppose A and B are 2×2 matrices with characteristic values λ_1 and λ_2, and γ_1 and γ_2 respectively where $|\lambda_1| \geq |\lambda_2|$ and $|\gamma_1| \geq |\gamma_2|$. Suppose ξ_1 and ξ_2 are the characteristic directions of A corresponding to λ_1 and λ_2 respectively, and $\xi_1 + \delta_1\xi_2$ and $\xi_2 + \delta_2\xi_1$ are the characteristic directions of B corresponding to γ_1 and γ_2 respectively, where $\bar{\xi}'_1\xi_1 = \bar{\xi}'_2\xi_2 = 1$ and ξ_1 and ξ_2 are linearly independent.

$$B(\xi_1 + \delta_1\xi_2) = \gamma_1(\xi_1 + \delta_1\xi_2).$$

Hence

$$B\xi_1 = \gamma_1\xi_1 + \delta_1\gamma_1\xi_2 - \delta_1 B\xi_2.$$

Similarly

$$B\xi_2 = \gamma_2\xi_2 + \delta_2\gamma_2\xi_1 - \delta_2 B\xi_1.$$

Therefore

$$B\xi_1 = (\gamma_1 - \delta_1\delta_2\gamma_2)\xi_1 + \delta_1(\gamma_1 - \gamma_2)\xi_2 + \delta_1\delta_2 B\xi_1.$$

If $\delta_1\delta_2 \neq 1$, then

$$B\xi_1 = \left(\frac{1}{1-\delta_1\delta_2}\right)\{(\gamma_1 - \delta_1\delta_2\gamma_2)\xi_1 + \delta_1(\gamma_1 - \gamma_2)\xi_2\}.$$

Also

$$B\xi_2 = \left(\frac{1}{1-\delta_1\delta_2}\right)\{(\gamma_2 - \delta_1\delta_2\gamma_1)\xi_2 + \delta_2(\gamma_2 - \gamma_1)\xi_1\}.$$

Suppose a characteristic vector of the product BA is of the form $\xi_1 + r\xi_2$, r is then a function of λ_i, γ_i, δ_i $(i = 1, 2)$. Suppose σ is the characteristic value of BA corresponding to $\xi_1 + r\xi_2$. Then

$$BA(\xi_1 + r\xi_2) = \sigma(\xi_1 + r\xi_2). \tag{4}$$

But

$$\begin{aligned} BA(\xi_1 + r\xi_2) &= B(\lambda_1\xi_1 + r\lambda_2\xi_2) \\ &= \left(\frac{1}{1-\delta_1\delta_2}\right)\{[\lambda_1(\gamma_1 - \delta_1\delta_2\gamma_2) + r\lambda_2\delta_2(\gamma_2 - \gamma_1)]\xi_1 \\ &\quad + [\lambda_1\delta_1(\gamma_1 - \gamma_2) + r\lambda_2(\gamma_2 - \delta_1\delta_2\gamma_1]\xi_2\}. \end{aligned} \tag{5}$$

Since ξ_1 and ξ_2 are linearly independent, it follows from (4) and (5) that

$$r = \frac{\lambda_1\delta_1(\gamma_1 - \gamma_2) + r\lambda_2(\gamma_2 - \delta_1\delta_2\gamma_1)}{\lambda_1(\gamma_1 - \delta_1\delta_2\gamma_2) + r\lambda_2\delta_2(\gamma_2 - \gamma_1)}.$$

Hence

$$\begin{aligned} f_1(r) \equiv \lambda_2\delta_2(\gamma_2 - \gamma_1)r^2 &+ \{\lambda_1(\gamma_1 - \delta_1\delta_2\gamma_2) - \lambda_2(\gamma_2 - \delta_1\delta_2\gamma_1)\}r \\ &+ \lambda_1\delta_1(\gamma_2 - \gamma_1) = 0. \end{aligned} \tag{6}$$

Similarly if $\xi_2 + t\xi_1$ is a characteristic vector for BA, then

$$\begin{aligned} f_2(t) \equiv \lambda_1\delta_1(\gamma_1 - \gamma_2)t^2 &+ \{\lambda_2(\gamma_2 - \delta_1\delta_2\gamma_1) - \lambda_1(\gamma_1 - \delta_1\delta_2\gamma_2)\}t \\ &+ \lambda_2\delta_2(\gamma_1 - \gamma_2) = 0. \end{aligned} \tag{7}$$

If only those matrices are studied for which λ_i, γ_i, ξ_i, δ_i are real, $|\delta_i| < 1$, $|\lambda_2/\lambda_1| < k < 1$ and $|\gamma_2/\gamma_1| < k < 1$, then bounds for the characteristic

directions of the product BA depend upon the relative signs of the determinants $|B| = \gamma_1\gamma_2$ and $|A| = \lambda_1\lambda_2$.

Case I. Suppose $\lambda_1\lambda_2 \geqq 0$ and $\gamma_1\gamma_2 \geqq 0$. Since $f_1(0) = \lambda_1\delta_1(\gamma_2 - \gamma_1)$ and $f_1(\delta_1) = \delta_1\gamma_2(1 - \delta_1\delta_2)(\lambda_1 - \lambda_2)$, the sign of $f_1(0)$ is the sign of $-\lambda_1\gamma_1\delta_1$ and that of $f_1(\delta_1)$ is the same as $\delta_1\lambda_1\gamma_2$. Since λ_1 and λ_2 are of the same sign, and γ_1 and γ_2 are of the same sign, there is a root of (6) between 0 and δ_1. Similarly, there is a root of (7) between 0 and δ_2.

Case II. If, however, $\lambda_1\lambda_2 \geqq 0$ and $\gamma_1\gamma_2 \leqq 0$, there exists a root of (7) between 0 and δ_2 but $f_1(0)$ and $f_1(\delta_1)$ are of the same sign. Consider

$$f_1(2\delta_1) = \delta_1[\lambda_1\gamma_1 + 2\lambda_2(-\gamma_2) - \lambda_1(-\gamma_2)] + \lambda_1\gamma_1K_1$$
$$|K_1| \leqq 4(k^2 + k)\,|\delta_1{}^2\delta_2|.$$

The term $[\lambda_1\gamma_1 + 2\lambda_2(-\gamma_2) - \lambda_1(-\gamma_2)]$ bears the sign of $\lambda_1\gamma_1$. Since in this case the sign of $\dfrac{f_1(0)}{\lambda_1\gamma_1}$ is the sign of $-\delta_1$, there is a root of (6) between 0 and $2\delta_1$ if the sign of $\dfrac{f_1(2\delta_1)}{\gamma_1\lambda_1}$ has the sign of δ_1. This is the case if the values of δ_1 and δ_2 are sufficiently small, so that K_1 will not affect the sign of $\dfrac{f_1(2\delta)}{\gamma_1\lambda_1}$. A bound for these values of δ_1 and δ_2 may be found depending upon k alone.

Case III. If $\lambda_1\lambda_2 \leqq 0$ and $\gamma_1\gamma_2 \geqq 0$, there exists a root of (6) between 0 and δ_1, but $f_2(0)$ and $f_2(\delta_2)$ are of the same sign.

$$f_2(-2\delta_2) = \delta_2[3\gamma_2(-\lambda_2) + 2\lambda_1\gamma_1 - \gamma_1(-\lambda_2)] + \lambda_1\gamma_1K_2$$
$$|K_2| \leqq 4(1 + 2k)\,|\delta_1\delta_2{}^2|.$$

The sign of $\dfrac{f_2(-2\delta_2)}{\lambda_1\gamma_1}$ is that of δ_2 if δ_1 and δ_2 are small enough so that K_2 does not affect the sign of $\dfrac{f_2(-2\delta_2)}{\lambda_1\gamma_1}$. Then there is a root of (7) between 0 and $-2\delta_2$ because the sign of $\dfrac{f_2(0)}{\lambda_1\gamma_1}$ is the sign of $-\delta_2$.

Case IV. If $\lambda_1\lambda_2 \leqq 0$ and $\gamma_1\gamma_2 \leqq 0$, consider

$$f_1\left(\frac{\delta_1}{1-k}\right) = \frac{\lambda_1\gamma_1\delta_1}{1-k}\left[1 - \frac{\lambda_2\gamma_2}{\lambda_1\gamma_1} + (1-k)\frac{\gamma_2}{\gamma_1} - (1-k)\right] + \gamma_1\lambda_1K_3$$
$$|K_3| \leqq \frac{3k - k^2}{(1-k)^2}\,|\delta_1{}^2\delta_2|.$$

The sign of $\dfrac{f_1\left(\dfrac{\delta_1}{1-k}\right)}{\lambda_1\gamma_1}$ is that of δ_1 if δ_1 and δ_2 are small enough so that K_3 does not affect the sign of $f_1\left(\dfrac{\delta_1}{1-k}\right)$. The sign of $\dfrac{f_1(0)}{\lambda_1\gamma_1}$ is that of $-\delta_1$, hence there is a root of (6) between 0 and $\dfrac{\delta_1}{1-k}$. The sign of $\dfrac{f_2(0)}{\lambda_1\gamma_1}$ is that of $-\delta_2$. Consider

$$f_2\left(\frac{-k\delta_2}{1-k}\right)=\frac{\lambda_1\gamma_1\delta_2}{1-k}\left\{k+\frac{\lambda_2}{\lambda_1}-k\frac{\lambda_2}{\lambda_1}-\frac{\lambda_2\gamma_2}{\lambda_1\gamma_1}\right\}+\lambda_1\gamma_1K_4$$

$$|K_4|\leq\frac{k^2(3-k)}{(1-k)^2}|\delta_1\delta_2{}^2|.$$

Neglecting K_4, the sign of $\dfrac{f_2\left(\dfrac{-k\delta_2}{1-k}\right)}{\lambda_1\gamma_1}$ is that of δ_2, hence there is a root of (7) between 0 and $\dfrac{-k\delta_2}{1-k}$ when δ_1 and δ_2 are sufficiently small, a bound for them depending upon the value of k alone.

Since the smaller roots of (6) and (7) are such that in Cases I, II, III, $|r|<|2\delta_1|$ and $|t|<|2\delta_2|$, and in Case IV $|r|<\dfrac{|\delta_1|}{1-k}$ and $|t|<\dfrac{k|\delta_2|}{1-k}$, when δ_1 and δ_2 are small quantities, the quadratic terms in r and t of (6) and (7) may be neglected as well as quadratic terms in δ_1 and δ_2. Consequently r is of the order of $\dfrac{\lambda_1\delta_1(\gamma_1-\gamma_2)}{\lambda_1\gamma_1-\lambda_2\gamma_2}$ and t is of the order of $\dfrac{\lambda_2\delta_2(\gamma_2-\gamma_1)}{\lambda_2\gamma_2-\lambda_1\gamma_1}$.

Equating coefficients in equations (4) and (5), the characteristic value corresponding to $\xi_1+r\xi_2$ becomes

$$\sigma=\frac{\lambda_1\gamma_1}{1-\delta_1\delta_2}\left\{1-\delta_1\delta_2\frac{\gamma_2}{\gamma_1}+r\delta_2\frac{\lambda_2}{\lambda_1}\left(\frac{\gamma_2}{\gamma_1}-1\right)\right\}. \tag{8}$$

Similarly the characteristic value ρ corresponding to $\xi_2+t\xi_1$ is

$$\rho=\frac{\lambda_2\gamma_2}{1-\delta_1\delta_2}\left\{1-\delta_1\delta_2\frac{\gamma_1}{\gamma_2}+t\delta_1\frac{\lambda_1}{\lambda_2}\left(\frac{\gamma_1}{\gamma_2}-1\right)\right\}. \tag{9}$$

$$\sigma=\frac{\lambda_1\gamma_1}{1-\delta_1\delta_2}\{1-K_5\}$$

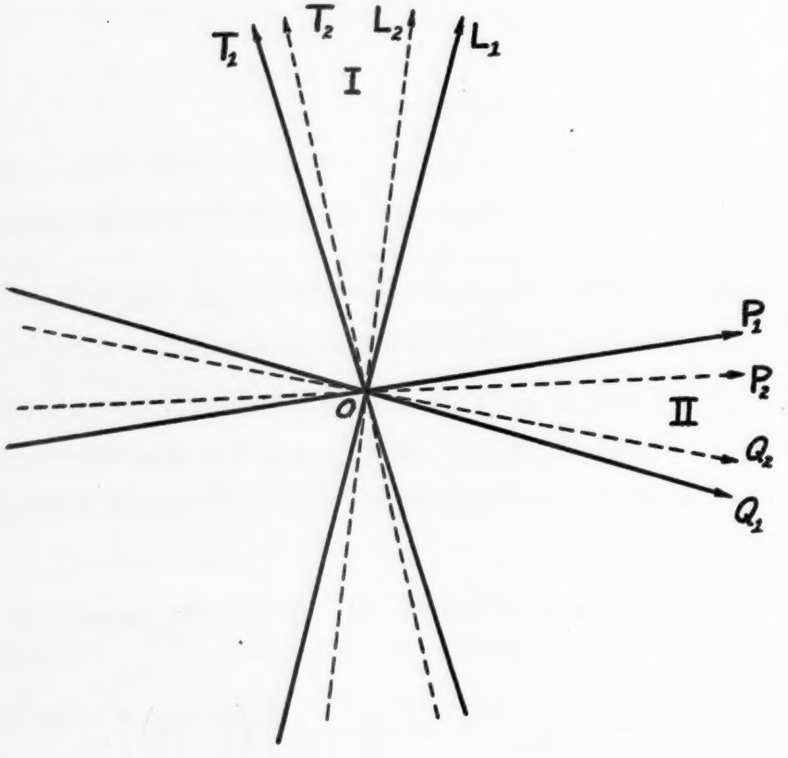
$$|K_5|<|\delta_1\delta_2|k+|r|\;|\delta_2|k(1-k).$$

Hence for small δ_1 and δ_2, σ is of the order of $\lambda_1\gamma_1$.

$$\frac{\rho}{\sigma}=\frac{\dfrac{\lambda_2\gamma_2}{\lambda_1\gamma_1}-\delta_1\delta_2\dfrac{\lambda_2}{\lambda_1}+t\delta_1\left(1-\dfrac{\gamma_2}{\gamma_1}\right)}{1-\delta_1\delta_2\dfrac{\gamma_2}{\gamma_1}+r\delta_2\dfrac{\lambda_2}{\lambda_1}\left(\dfrac{\gamma_2}{\gamma_1}-1\right)}$$

$$=\frac{\dfrac{\lambda_2\gamma_2}{\lambda_1\gamma_1}-K_6}{1-K_5}$$

$$|K_6|<|\delta_1\delta_2|k+(1+k)\,|t|\;|\delta_1|.$$



Hence $\dfrac{\rho}{\sigma}$ is of the order of $\dfrac{\lambda_2\gamma_2}{\lambda_1\gamma_1}$ for small δ_1 and δ_2 from which it follows that ρ is of the order of $\lambda_2\gamma_2$.

The nature of the results in Cases I-IV were arrived at geometrically. The simplest case is illustrated by the accompanying figure. Let OT_1 and OP_1 be the directions ξ_1 and ξ_2 corresponding to the larger and smaller roots of A, λ_1 and λ_2 respectively. Let OL_1 and OQ_1 be the directions $\xi_1+\delta_1\xi_2$ and

$\xi_2 + \delta_2\xi_1$, corresponding to the larger and smaller roots of B, γ_1 and γ_2 respectively. In this case $\lambda_1 > \lambda_2 > 0$ and $\gamma_1 > \gamma_2 > 0$. Under the transformation with matrix A, OT_1 and OP_1 are invariant but OL_1 and OQ_1 move into regions I and II, along the directions OL_2 and OQ_2. Under the transformation with matrix B, OT_1 and OP_1 move into regions I and II along the directions OT_2 and OP_2. From continuity considerations it follows that invariant directions for the product BA lie in regions I and II. In Cases II-III and less simple cases of I geometric arguments were also helpful but the diagrams were complicated by reflection caused by the occurrence of negative λ_i and γ_i.

THEOREM 6. *If $\{A_i\}$ is a sequence of second order matrices with the characteristic values of A_i equal to λ_{i1} and λ_{i2}, where* $\left|\dfrac{\lambda_{i2}}{\lambda_{i1}}\right| < k < 1$, *if ξ_1 and ξ_2 are two directions in space, then for every arbitrarily chosen number $\varepsilon > 0$ there exist numbers $\delta_{\epsilon k} > 0$ and $\delta'_{\epsilon k} > 0$ such that if the characteristic directions corresponding to λ_{i1} and λ_{i2} lie in the ranges $\xi_1 \pm \delta_{\epsilon k}\xi_2$ and $\xi_2 \pm \delta'_{\epsilon k}\xi_1$ respectively, then the characteristic directions of any product of a finite sequence of these matrices lie in the ranges $\xi_1 \pm \epsilon\xi_2$ and $\xi_2 \pm \epsilon\xi_1$. It is assumed that all quantities under consideration are real.*

Suppose it is known that for each of a sequence of matrices the characteristic directions corresponding to λ_{i1} and λ_{i2} lie within the ranges $\xi_1 \pm \delta_1\xi_2$ and $\xi_2 \pm \delta_2\xi_1$ respectively. Let $\delta_1 > 0$ and $\delta_2 > 0$. The product Q_m of m matrices may be expressed in the form

$$Q_m = N_m \cdot P_{k+1} \cdot M_k \cdot \cdot \cdot M_2 \cdot P_2 \cdot M_1 \cdot P_1,$$

where P_j is a product of matrices A_i each with positive determinant; M_j is a product of matrices A_i with the determinants of the first and last matrices negative, and those determinants of the intervening matrices positive. Let $M_j = A_{j1}T_jA_{j2}$. The determinants of T_j and M_j are positive. N_m is either the identity or else is a product of matrices A_i with only the right-hand factor having a negative determinant, all others being positive. Any P_j or T_j may be equal to the identity.

From Case I, for every P_j and T_j the characteristic directions corresponding to the larger and smaller characteristic values lie in the ranges $\xi_1 \pm \delta_1\xi_2$ and $\xi_2 \pm \delta_2\xi_1$ respectively. When δ_1 and δ_2 are properly restricted, the characteristic values of the product of t of the matrices A_i are approximately $\prod_{i=1}^{t} \lambda_{i1}$ and $\prod_{i=1}^{t} \lambda_{i2}$. Then $\left|\dfrac{\prod_{i=1}^{t} \lambda_{i1}}{\prod_{i=1}^{t} \lambda_{i2}}\right| < k < 1$. It follows from Case III that

the characteristic direction corresponding to the larger characteristic value of $T_j A_{j2}$ lies in $\xi_1 \pm \delta_1 \xi_2$. Using the above theory of Cases I-IV, to compute bounds for the invariant direction corresponding to the smaller characteristic value it is only necessary to establish limits when the characteristic directions of T_j and A_{j2} are on the boundaries of the regions, $\xi_1 \pm \delta_1 \xi_2$ and $\xi_2 \pm \delta_2 \xi_1$. For example, suppose that the invariant directions of A_{j2} are $\xi_1 - \delta_1 \xi_2$ and $\xi_2 + \delta_2 \xi_1$ and the invariant directions of T_j are $\xi_1 + \delta_1 \xi_2$ and $\xi_2 - \delta_2 \xi_1$. To apply the results of Case III note that $\xi_1 + \delta_1 \xi_2$ and $\xi_2 - \delta_2 \xi_1$ lie in the directions

$$(\xi_1 - \delta_1 \xi_2) + \frac{2\delta_1}{1 - \delta_1 \delta_2} (\xi_2 + \delta_2 \xi_1)$$

and

$$(\xi_2 + \delta_2 \xi_1) - \frac{2\delta_2}{1 - \delta_1 \delta_2} (\xi_1 - \delta_1 \xi_2)$$

respectively. Then the invariant direction of the product corresponding to the smaller characteristic value lies between $\xi_2 + \delta_2 \xi_1$ and

$$(\xi_2 + \delta_2 \xi_1) + \frac{4\delta_2}{1 - \delta_1 \delta_2} (\xi_1 - \delta_1 \xi_2),$$

that is between $\xi_2 + \delta_2 \xi_1$ and $\xi_2 + \dfrac{5 - \delta_1 \delta_2}{1 - 5\delta_1 \delta_2} \delta_2 \xi_1$. Similarly, if the other possible bounds for the invariant directions of A_{j2} and T_j are considered, it follows that the direction of the product $T_j A_{j2}$ is bounded by $\xi_2 \pm \mu_2 \delta_2 \xi_1$ where $\mu_2 = \left| \dfrac{5 - \delta_1 \delta_2}{1 - 5\delta_1 \delta_2} \right|$; μ_2 is of the order of 5 if δ_1 and δ_2 are small.

Bounds for the characteristic directions of $M_j = A_{j1} T_j A_{j2}$ are given through Case IV. For example, suppose the invariant direction corresponding to the larger root of $T_j A_{j2}$ is $\xi_1 - \delta_1 \xi_2$, and that of A_{j1} is $\xi_1 - \delta_1 \xi_2$. Suppose the invariant direction corresponding to the smaller root of $T_j A_{j2}$ is either of $\xi_2 \pm \mu_2 \delta_2 \xi_1$. To apply the results of Case IV, the direction $\xi_1 - \delta_1 \xi_2$ must be expressed in the form

$$\xi_1 + \delta_1 \xi_2 - \frac{2\delta_1}{1 \pm \mu_2 \delta_1 \delta_2} (\xi_2 \pm \mu_2 \delta_2 \xi_1).$$

It follows that the characteristic direction corresponding to the larger characteristic value of $A_{j1} T_j A_{j2}$ lies between $\xi_1 + \delta_1 \xi_2$ and $\xi_1 - \tau_1 \delta_1 \xi_2$ where τ_1 is either

$$\frac{(1+k) - (1-k)\delta_1 \delta_2 \mu_2}{(1-k) - (1+k)\delta_1 \delta_2 \mu_2}$$

or

$$\frac{(1+k) + (1-k)\delta_1 \delta_2 \mu_2}{(1-k) + (1+k)\delta_1 \delta_2 \mu_2}.$$

If the other bounds for the characteristic directions of A_{j1} and T_jA_{j2} are considered, bounds for the characteristic directions of M_j are similarly found to be $\xi_1 \pm \tau_1\delta_1\xi_2$ and $\xi_2 \pm \tau_2\delta_2\xi_1$. The orders of τ_1 and τ_2 are less than $\frac{6}{1-k}$. Since the determinant of M_j is positive, the characteristic directions of M_jP_j lie within the larger of the two ranges given for M_j and P_j and since these ranges are independent of j, the characteristic directions of any number of products of M_jP_j lie in the same range. It is assumed that δ_1 and δ_2 are sufficiently small so that the approximation formulas hold to within the limits necessary for the above reasoning, and so that the characteristic values of a product of matrices closely approximate the product of the characteristic values. If an odd number of matrices with negative determinants is included in the product Q_m, that is, if the determinant of N_m is negative, the characteristic direction corresponding to the larger characteristic value of Q_m may fall outside of $\xi_1 \pm \tau_1\delta_1\xi_2$, but since the characteristic directions of N_m according to Case III are in the ranges $\xi_1 \pm \delta_1\xi_2$ and $\xi_2 \pm \mu_2\delta_2\xi_1$, it can be shown from Case II that the characteristic directions of Q_m lie in $\xi_1 \pm \sigma_1\delta_1\xi_2$ and $\xi_2 \pm \tau_2\delta_2\xi_1$. The bound $\xi_1 \pm \sigma_1\delta_1\xi_2$ is of the order of $\xi_1 + \frac{3-k}{1-k}\delta_1\xi_2$.

Hence if ε is an arbitrary number, sufficient conditions for the proof of Theorem 6 are that $\delta_{\epsilon k}$ and $\delta'_{\epsilon k}$ are sufficiently small to satisfy the conditions of Cases I-IV which depend upon k alone and also $\delta_{\epsilon k} < \frac{\varepsilon}{\sigma_1}$ and $\delta'_{\epsilon k} < \frac{\varepsilon}{\tau_2}$.

UNIVERSITY OF WISCONSIN.

THE INTERRELATIONS OF THE FUNDAMENTAL SOLUTIONS OF THE HYPERGEOMETRIC EQUATION; LOGARITHMIC CASE.*

By Lyle E. Mehlenbacher.

1. The problem of this paper is to study the exact nature of the linear relations existing between the fundamental solutions of the Hypergeometric Differential Equation

$$z(1-z)\frac{d^2y}{dz^2}+[\gamma-(\alpha+\beta+1)z]\frac{dy}{dz}-\alpha\beta y=0 \tag{1}$$

in which we may consider the constants α, β, γ and the variable z as real or complex. The relations of this kind which exist in case each of the three regular singular points $z=0$, $z=1$, $z=\infty$ is non-logarithmic in character have already been considered by Forsyth,[1] Lindelöf [2] and Barnes.[3] What is proposed here is to determine the linear relations between the fundamental solutions of (1) when one or more of these solutions becomes logarithmic in character. We shall, for the sake of brevity, actually determine only those linear relations existing between the fundamental solutions about the point $z=0$ and the solutions about the point $z=\infty$ when one of the solutions about the latter point is logarithmic. The methods employed are readily applied to the cases in which either the point $z=0$, or the point $z=1$ is logarithmic. The complete set of interrelations in both the non-logarithmic cases and the logarithmic cases are included in the author's dissertation written at the University of Michigan under the direction of Professor W. B. Ford, and published in pamphlet form by Edwards Brothers of Ann Arbor, Michigan.

2. For purposes of future reference we shall first set down some of the more important relations which occur in obtaining the fundamental solutions

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[1] A. R. Forsyth, *A Treatise on Differential Equations*, 3rd ed., London, Macmillan and Co. (1903), pp. 203-222.

[2] Ernst Lindelöf, "Sur l'Intégration de l'Équation de Kummer," *Acta Societatis Fennicae*, (1), Tome 19 (1893), pp. 3-31.

[3] E. W. Barnes, "A new development of the theory of the hypergeometric functions," *Proceedings of the London Mathematical Society*, Series 2, vol. 6 (1908), pp. 141-177.

of (1) about each of its three regular singular points $z=0$, $z=1$, $z=\infty$ by the usual methods of the Fuchs[4] theory.

The indicial equation of (1) corresponding to the singular point $z=0$ is

$$k(k-1)+\gamma k=0, \tag{2}$$

the roots of which are found to be

$$k_1=0, \qquad k_2=1-\gamma. \tag{3}$$

Assuming that k_1-k_2 is non-integral, we shall therefore have, according to the Fuchs theory two fundamental solutions Y_1 and Y_2 about the point $z=0$ having the forms

$$Y_1=z^{k_1}\sum_{n=0}^{\infty}g_1(n)z^n; \qquad Y_2=z^{k_2}\sum_{n=0}^{\infty}g_2(n)z^n. \tag{4}$$

Moreover, if we put

$$f(z,k)=k(k-1)(1-z)+k[\gamma-(\alpha+\beta+1)z]-\alpha\beta z,$$

and

$$f_0(k)=[f(z,k)]_{z=0}=k(k-1)+\gamma k,$$

$$f_1(k)=\left[\frac{\partial f(z,k)}{\partial z}\right]_{z=0}=-k(k-1)-(\alpha+\beta+1)\,k-\alpha\beta,$$

then the same theory indicates that $g_1(n)$ will satisfy the linear recurrence relation

$$g_1(n)f_0(k+n)+g_1(n-1)f_1(k+n-1)=0 \tag{5}$$

in which k is given the value $k_1=0$; while $g_2(n)$ will satisfy the same relation with the value $k_2=1-\gamma$ used for k.

The values of $g_1(0)$, $g_2(0)$ are arbitrary. For definiteness, let us take $g_1(0)=g_2(0)=1$, in which case $g_1(n)$, $g_2(n)$ become completely determined by (5) for $n=1,2,3,\cdots$.

When the $g_1(n)$, $g_2(n)$ are thus determined we find that when $k_1-k_2=\gamma-1$ is non-integral the two fundamental solutions about the singular point $z=0$ can be put in the forms

$$Y_1=F(\alpha,\beta,\gamma;z), \quad Y_2=z^{1-\gamma}F(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma;z), \tag{6}$$

where the $F(\alpha,\beta,\gamma;z)$ represents the well-known Hypergeometric Function defined by

$$F(\alpha,\beta,\gamma;z)$$
$$=1+\sum_{n=1}^{\infty}\frac{\alpha(\alpha+1)\cdots(\alpha+n-1)\beta(\beta+1)\cdots(\beta+n-1)}{1\cdot2\cdot3\cdots n\gamma(\gamma+1)\cdots(\gamma+n-1)}z^n.$$

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[4] See J. Horn, *Gewöhnliche Differentialgleichungen beliebiger Ordnung*, Sammlung Schubert L., Leipzig (1905), Section 34.

The indicial equation of (1) corresponding to the point $z = \infty$ is found to be

$$k(k-1) - (\alpha + \beta - 1)\,k + \alpha\beta = 0, \tag{7}$$

the roots of which are

$$k'_1 = \alpha, \qquad k'_2 = \beta, \tag{8}$$

after which we may show that if $k'_1 - k'_2 = \alpha - \beta$ is non-integral the two fundamental solutions of (1) about the singular point $z = \infty$ are

$$\begin{aligned} Y_5 &= z^{-\alpha}F(\alpha, \alpha - \gamma + 1, \alpha - \beta + 1; 1/z), \\ Y_6 &= z^{-\beta}F(\beta, \beta - \gamma + 1, \beta - \alpha + 1; 1/z). \end{aligned} \tag{9}$$

When the exponents (8) differ by an integer, that is when

$$\alpha - \beta = n = \text{an integer} \geq 0, \tag{10}$$

the point $z = \infty$ becomes a so-called logarithmic point. The fundamental solutions about $z = \infty$ then take the forms

$$\begin{aligned} &Y_5 = \text{as defined in (9)}, \\ &\bar{Y}_6 = Y_5 \operatorname{Log}(1/z) + z^{-k'_2} \sum_{v=0}^{\infty} h_v z^{-v}; \; h_n = 0, h_0 \neq 0 \text{ unless } n = 0. \end{aligned} \tag{11}$$

3. It is our purpose in what follows to determine the exact forms of the linear relations which connect Y_1 with Y_5 and $\bar{Y}_6$; likewise those connecting Y_2 with the same Y_5, $\bar{Y}_6$. The methods which we shall use rest upon certain general results obtained by Professor Ford in Chapter I of his recent book entitled *The Asymptotic Developments of Functions Defined by Maclaurin Series.*[5] In particular, we shall employ the following General Theorem there established:

"THEOREM. *If the coefficients $g(n)$ of the power series*

$$f(z) = \sum_{n=0}^{\infty} g(n) z^n; \; \textit{radius of convergence} > 0, \tag{A}$$

may be considered as a function $g(w)$ of the complex variable $w = x + iy$ and as such satisfies the two following conditions when considered throughout any arbitrary right half plane $x > x_0$:

(a) *is single-valued and analytic,*

(b) *is such that for all $|y|$ sufficiently large one may write*

$$|g(x + iy)| < Ke^{\epsilon|y|} \tag{B}$$

[5] *Michigan Science Series*, vol. 11 (1936).

where ϵ is an arbitrarily small positive quantity given in advance and where K depends only upon x_0 and ϵ, then the function $f(z)$ defined by (A) is analytic throughout any sector S (vertex at origin) of the z-plane which does not include the positive half of the real axis and $f(z)$ within S is developable asymptotically as follows:

$$\text{(C)} \qquad f(z) \sim -\sum_{n=0}^{\infty} \frac{g(n)}{z^n}.\text{''}$$

We shall make use also of the following Remarks (b) and (e) relative to the foregoing theorem, as noted in the same work.

"(b) In case conditions (a) and (b) of the theorem are satisfied except that $g(w)$ has p $(p \geq 1)$ singularities situated at the points $w = w_1, w_2, w_3, \cdots, w_p$, none of which are negative integers, the theorem continues to hold true provided one subtracts from the right member of (C) the sum of p loop integrals of the function

$$\text{(D)} \qquad \frac{g(w)(-z)^w}{2i \operatorname{Sin} \pi w}, \cdots.$$

"In case the singular points $w = w_m$ are poles, the loop integrals may evidently be replaced by integrations of (D) over small circles, so that in such cases the theorem continues to hold, provided that one subtracts from the right member of (C) the sum of the residues of the function

$$\text{(E)} \qquad \frac{\pi g(w)(-z)^w}{\operatorname{Sin} \pi w}$$

at the various poles $w = w_m$.

"(e) The theorem may be applied to any Maclaurin series (A) in which $g(w)$, besides satisfying condition (a), is such that we may write, when $x > x_0$ and $|w|$ is large, $|g(w)| < K|w|^c$, where K and c are constants of which the latter may be positive, negative or zero."

Whenever we apply Remark (b) it shall be understood that for any given value of $z = \rho e^{i\phi}$ the function $(-z)^w$ is rendered precise in meaning through the following convention:

$$(12) \qquad (-z)^w = e^{w \log(-z)} = e^{w\,[\log \rho + i(\phi - \pi)]} = z^w e^{-i\pi w}; \qquad 0 \leq \phi < 2\pi.$$

4. Theorem. Employing the above results, we proceed to establish the following theorem which, so far as we have been able to determine, is new:

THEOREM. *The solutions Y_1 and Y_2 defined in* (6), *when extended analytically outside their circle of convergence, may be expressed linearly in*

terms of the solutions Y_5 *and* $\bar{Y}_6$, *defined in* (11), *in the following forms, it being understood throughout that* $0 < \arg z < 2\pi$:

$$Y_1 = \frac{\Gamma(\gamma)[\psi(\alpha) + \psi(\gamma - \alpha) + 2c + i\pi - \sum_1^n (1/s)]e^{i\pi(\beta+1)}}{\Gamma(\beta)\Gamma(\gamma - \alpha)\Gamma(n+1)} Y_5 + \frac{\Gamma(\gamma)e^{i\pi(\beta+1)}}{\Gamma(\beta)\Gamma(\gamma-\alpha)\Gamma(n+1)} \bar{Y}_6$$

$$Y_2 = \frac{\Gamma(2-\gamma)[\psi(\alpha - \gamma + 1) + \psi(1-\alpha) + 2c + i\pi - \sum_1^n (1/s)]e^{i\pi(\beta-\gamma)}}{\Gamma(\beta - \gamma + 1)\Gamma(1-\alpha)\Gamma(n+1)} Y_5 + \frac{\Gamma(2-\gamma)e^{i\pi(\beta-\gamma)}}{\Gamma(\beta-\gamma+1)\Gamma(1-\alpha)\Gamma(n+1)} \bar{Y}_6$$

where $\psi(\alpha)$ *is the well-known Psi-function defined as the logarithmic derivative of the Gamma-function, that is* $\psi(\alpha) = \Gamma'(\alpha)/\Gamma(\alpha)$, *where* C *denotes Euler's constant defined by* $C = -\psi(1)$, *and where* $\sum_1^n (1/s)$ *becomes* 0 *when* $n = 0$.[6]

Proof. Returning to the recurrence relation (5), we substitute the values of $f_0(k+n)$ and $f_1(k+n-1)$ and, in order that the notation shall conform to that which is customary in the theory of difference equations, we replace $g(n)$, $g(n-1)$ respectively by $u(x)$, $u(x-1)$ and subsequently advance x to $x+1$. The relation (5) thus becomes

$$q_1(x)u(x+1) + q_0(x)u(x) = 0, \tag{13}$$

where

$$q_1(x) = (k+x+1)(k+x) + \gamma(k+x+1),$$
$$q_0(x) = -(k+x)(k+x-1) - (\alpha+\beta+1)(k+x) - \alpha\beta,$$

and k takes the value k_1 or k_2 of (3).

For those particular values of k in which we are interested, namely $k = k_1$ and $k = k_2$ of (3), the roots of $q_1(x) = 0$ are found to be

$$x = -1, \qquad x = -(2k+\gamma),$$

and the roots of $q_0(x) = 0$ are

$$r_1 = -\alpha - k, \qquad r_2 = -\beta - k. \tag{14}$$

Hence we may write

$$q_1(x) = (x+1)(x+2k+\gamma), \qquad q_0(x) = -(x-r_1)(x-r_2).$$

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[6] See L. M. Milne-Thomson, *The Calculus of Finite Differences*, London, Macmillan and Co. (1933), p. 245 and p. 250.

When we substitute these values into (13) and then solve the resulting first order difference equation by elementary methods we obtain as the particular solution $u(x)$ which takes the value 1 when $x=0$

$$u(x)=g(x)=\frac{\Gamma(2k+\gamma)\Gamma(x-r_1)\Gamma(x-r_2)}{\Gamma(-r_1)\Gamma(-r_2)\Gamma(x+1)\Gamma(x+2k+\gamma)}, \tag{15}$$

in which we shall assume at first that neither r_1 nor r_2 is zero or a positive integer, thus rendering $u(x)\not\equiv 0$.

We must notice here that since $\alpha-\beta=n=$ an integer ≥ 0, it follows from (14) that $r_1=r_2-n$.

Now, each of the fundamental solutions Y_1, Y_2 is of the form

$$Y=z^k\sum_{n=0}^{\infty} g(n)z^n; \qquad g(0)=1, \tag{16}$$

where the $g(n)$ is defined by (15). The function $g(x)$ in (15) coincides with the $g(n)$ of (16) when $x=0,1,2,3,\cdots$ and is, moreover, analytic throughout the finite x-plane except for poles of the first order at the points

$$x=r_2, r_2-1, r_2-2, \cdots, r_2-n+1 \tag{17}$$

and poles of the second order at

$$x=r_1, r_1-1, r_1-2, \cdots, \tag{18}$$

while in distant portions of the plane lying in any arbitrary right half-plane, the same function $g(x)$ satisfies the condition described in Remark (e) quoted above.

In order to determine the asymptotic behavior of the function $Y_1(x)$ defined by the Maclaurin series (16) we may therefore apply the General Theorem quoted above, subject to Remark (b). When we do so, we see that in our present case the $g(-x)$; $x=1,2,\cdots$, in formula (C) all vanish owing to the general relation $1/\Gamma(-x)=0$; $x=1,2,3,\cdots$. Furthermore, Remark (b) requires that we subtract from the right-hand side of (C) the sum of the residues of the function

$$\begin{aligned}\frac{\pi g(x)(-z)^x}{\operatorname{Sin}\pi x}&=\Gamma(x)\Gamma(1-x)g(x)(-z)^x\\ &=-\frac{\Gamma(-x)\Gamma(2k+\gamma)\Gamma(x-r_1)\Gamma(x-r_2)}{\Gamma(-r_1)\Gamma(-r_2)\Gamma(x+2k+\gamma)}(-z)^x\end{aligned} \tag{19}$$

at its poles (17) and (18). However, inasmuch as we are interested only in determining the values of the constants which join the solutions Y_1 and Y_2

linearly with the solutions Y_5 and $\bar{Y}_6$ it is sufficient for our purpose to determine the residues of (19) at the two poles $x = r_1$ and $x = r_2$.

The residue of (19) at the first of the points (17) involves z to the power $-\beta - k$ and the residues at the remaining points of the same set involve z to the lower powers $-\beta - k - 1$ to $-\beta - k - n + 1 = -\alpha - k + 1$. It will be shown presently that the residue at the first of the poles (18) involves $z^{-\alpha-k}$ and also $z^{-\alpha-k}\log z$, while the residues at the remaining points (18) involve $z^{-\alpha-k-s}$ and $z^{-\alpha-k-s}\log z$ $(s = 1, 2, 3, \cdots)$. It follows that the coefficient of the highest power of z from both sources will come from the residue at the first point of (17). In order to determine the linear relations which we are seeking it suffices then to determine the residue of (19) at the point $x = r_2$ and the logarithmic part of the residue of (19) at the point $x = r_1$. The residue at the pole $x = r_2$ is found by an elementary theorem in the calculus of residues to be

$$(20) \qquad -\frac{\Gamma(2k+\gamma)\Gamma(r_2 - r_1)}{\Gamma(-r_1)\Gamma(r_2 + 2k + \gamma)}(-z)^{r_2}.$$

When we substitute into (20) the values of r_1 and r_2 as defined, and apply the convention (12) to $(-z)^{r_2}$ this residue may be written as

$$(21) \qquad -C_2(k, \alpha, \beta, \gamma)z^{-\beta-k} = -\frac{\Gamma(2k+\gamma)\Gamma(n)e^{i\pi(\beta+k)}z^{-\beta-k}}{\Gamma(\alpha+k)\Gamma(\gamma-\beta+k)}.$$

In order to obtain the residue of (19) at its pole of the second order $x = r_1$ we write (19) in the form $-\dfrac{M(x)z^x}{Q^2(x)}$, where

$$M(x) = \frac{\Gamma(-x)\Gamma(2k+\gamma)\Gamma^2(x - r_1 + 1)e^{-i\pi x}}{\Gamma(-r_1)\Gamma(-r_2)\Gamma(x+2k+\gamma)(x-r_1-1)(x-r_1-2)\cdots(x-r_1-n)},$$

$$(22) \qquad Q(x) = x - r_1,$$

The residue of (19) at $x = r_1$ is now found to be

$$(23) \qquad -M'(r_1)z^{r_1} - M(r_1)z^{r_1}\log z.$$

As we have explained above, we have at first only to determine the logarithmic part of this residue, which by reference to (22) is easily found to be

$$(24) \quad -C_1(k, \alpha, \beta, \gamma)z^{-\alpha-k}\log z = \frac{\Gamma(2k+\gamma)e^{i\pi(\beta+k)}z^{-\alpha-k}}{\Gamma(\beta+k)\Gamma(k+\gamma-\alpha)\Gamma(n+1)}\log z.$$

We may now apply the General Theorem previously quoted together with its accompanying remarks. In this way we arrive at the following preliminary result:

The function Y defined in (16), when extended analytically outside its circle of convergence, may be developed asymptotically in the following form, it being understood throughout that $0 < \arg z < 2\pi$:

$$(25)\qquad Y \sim C_1(k,\alpha,\beta,\gamma)z^{-\alpha}[1+(\ )/z+(\ )/z^2+\cdots]\log z + C_2(k,\alpha,\beta,\gamma)z^{-\beta}[1+(\ )/z+(\ )/z^2+\cdots]$$

where $C_1(k,\alpha,\beta,\gamma)$ and $C_2(k,\alpha,\beta,\gamma)$ are defined in (24) and (21) respectively. The asymptotic expansion of the solution Y_1 of (1) results from this when the value $k = k_1 = 0$ is used, while the corresponding expansion for Y_2 results likewise from the use of $k_2 = 1 - \gamma$.

We are concerned here, however, with the expression of the function Y_1 as a linear combination of the solutions Y_5 and $\bar{Y}_6$, that is, we wish to determine constants K_1 and K_2 such that

$$Y_1 \sim K_1 Y_5 + K_2 \bar{Y}_6$$

with a similar expression for Y_2. Upon referring to the definitions (11) of Y_5 and $\bar{Y}_6$, we can identify by (25) the constant K_2 with $-C_1(k,\alpha,\beta,\gamma)$. The solution Y_5 contains the factor $z^{-\alpha} = z^{-\beta-n}$. We know from the Fuchs theory that the non-logarithmic part of the solution $\bar{Y}_6$ does not involve z to this power since $h_n = 0$. We may therefore identify the constant K_1 with the coefficient of $z^{-\beta-n}$ in the non-logarithmic part of the right side of (25). This coefficient, before taking out the factor $C_2(k,\alpha,\beta,\gamma)$, is the value of the coefficient of $z^{r_1} = z^{-\alpha-k}$ in the non-logarithmic part of the residue (23). This is easily found, by use of (22), to be

$$(26)\quad C_n(k,\alpha,\beta,\gamma) = \frac{\Gamma(2k+\gamma)[\psi(\alpha+k)+\psi(\gamma-\alpha+k)+2C+i\pi-\sum_1^n(1/s)]e^{i\pi(\beta+k+1)}}{\Gamma(\beta+k)\Gamma(\gamma-\alpha+k)\Gamma(n+1)}.$$

We thus arrive at the following result for the solution Y_1:

The solution Y_1, when extended analytically outside its circle of convergence, may be developed asymptotically in the following form, it being understood throughout that $0 < \arg z < 2\pi$:

$$(27)\qquad Y_1 \sim C_n(k,\alpha,\beta,\gamma)Y_5 - C_1(k,\alpha,\beta,\gamma)\bar{Y}_6$$

where $C_n(k,\alpha,\beta,\gamma)$ and $C_1(k,\alpha,\beta,\gamma)$ are defined in (26) and (24) respectively and in which k takes the value $k_1 = 0$.

The corresponding result expressing the solution Y_2 linearly in terms of Y_5 and $\bar{Y}_6$ is obtained from the foregoing by the use of $k = k_2 = 1 - \gamma$ for k.

We note here that the constants $C_1(k,\alpha,\beta,\gamma)$, $C_2(k,\alpha,\beta,\gamma)$ and

$C_n(k,\alpha,\beta,\gamma)$ all preserve meanings regardless of whether $r_1 = -\alpha - k$ or $r_2 = -\beta - k$ is zero or a positive integer, so that the restriction made in (15) may now be removed. Since the series Y_5 and $\bar{Y}_6$ are known to be convergent for the indicated values of z, the symbol $\sim$ may be changed to $=$. We have now only to introduce into (27) the values of the constants $C_n(k,\alpha,\beta,\gamma)$ and $C_1(k,\alpha,\beta,\gamma)$ as defined in order to arrive at the final results stated in the Theorem.

5. Further results. The solutions of (1) about the point $z = 1$ are

$$Y_3 = \mathrm{F}(\alpha,\beta,\alpha+\beta-\gamma+1;1-z),$$
$$Y_4 = (1-z)^{\gamma-\alpha-\beta}F(\gamma-\alpha,\gamma-\beta,\gamma-\alpha-\beta+1;1-z).$$

In order to determine the linear relationships between the solutions Y_3 and Y_4 and the solutions Y_5 and $\bar{Y}_6$ we first make the transformation $z = 1 - z'$ in (1) and subsequently apply the same methods to this transformed equation as we have employed in the determination of the results in Theorem I. After these results have been obtained, we make the reverse transformation $z' = 1 - z$. The final result obtained in this manner is stated in the following theorem:

THEOREM II. *The solutions Y_3 and Y_4, when extended analytically outside their circle of convergence, may be expressed linearly in terms of Y_5 and $\bar{Y}_6$ in the following forms, it being understood throughout that* $0 < \arg(1-z) < 2\pi$:

$$Y_3 = \frac{\Gamma(\alpha+\beta-\gamma+1)[\psi(\alpha)+\psi(\beta-\gamma+1)+2c+i\pi-\sum_1^n(1/s)]e^{i\pi(\beta+1)}}{\Gamma(\beta)\Gamma(\beta-\gamma+1)\Gamma(n+1)}Y_5$$
$$+\frac{\Gamma(\alpha+\beta-\gamma+1)e^{i\pi(\beta+1)}}{\Gamma(\beta)\Gamma(\beta-\gamma+1)\Gamma(n+1)}\bar{Y}_6$$

$$Y_4 = \frac{\Gamma(\gamma-\alpha-\beta+1)[\psi(\gamma-\beta)+\psi(1-\alpha)+2c+i\pi-\sum_1^n(1/s)]e^{i\pi(\gamma-\alpha+1)}}{\Gamma(\gamma-\alpha)\Gamma(1-\alpha)\Gamma(n+1)}Y_5$$
$$+\frac{\Gamma(\gamma-\alpha-\beta+1)e^{i\pi(\gamma-\alpha+1)}}{\Gamma(\gamma-\alpha)\Gamma(1-\alpha)\Gamma(n+1)}\bar{Y}_6$$

When the fundamental solutions about either the singular point $z = 0$ or the singular point $z = 1$ are logarithmic the resulting interrelations are obtained by first making the appropriate transformation of the independent variable in (1), employing the same procedure as we have used above in the proof of Theorem I, and then making the reverse transformation in the results.

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THE THEOREMS OF GAUSS-BONNET AND STOKES.*

By E. R. van Kampen.

The idea of a parallel displacement in vector analysis opened new possibilities also for the differential geometry of surfaces. The present note contains an elementary proof of the Gauss-Bonnet theorem based on this idea. A systematic use is made of the correspondence between the surface and the spaces of its parameter systems for purposes of subdivision in **7** and for the determination of the variation of an angle in **4**.

Third continuous derivatives of the parametric equations of the surface occur only at the beginning of **5**. Their use may be eliminated by the short additional consideration in **6**. Thus one obtains also for the *Theorema Egregium* (without the explicit expression for the Gaussian curvature) a proof for surfaces of class C_2. It may be of interest that the proof below makes no use either of the second or of the first fundamental form.

In **10** a short proof of Stokes theorem is given which is valid for a vector-field of class C_1 and a compact region T on a surface of class C_1 if the boundary of T consists of rectifiable arcs.

1. Let the letters $\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}, \cdots$ represent vectors in 3-space and let $\boldsymbol{u} \cdot \boldsymbol{v}$, $\boldsymbol{u} \times \boldsymbol{v}$, $(\boldsymbol{uvw}) = \boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w})$ represent the scalar product, vector product and triple product of vectors respectively. Let a surface S be given by a parametric representation of the form

$$\boldsymbol{x} = \boldsymbol{x}(u^1, u^2), \tag{1}$$

defined in the whole plane of the scalar parameters u^1, u^2. The letters R and T will be used to designate curves and regions of S and at the same time the corresponding curves and regions of the u^i-plane. The label i will have the range 1, 2 and any term containing such a label both as a subscript and as a superscript must be summed over this range. It will be supposed that (1) is at least of class C_1 and that

$$(\boldsymbol{x}_1 \times \boldsymbol{x}_2) \neq 0, \tag{2}$$

where the subscripts 1 and 2 represent partial differentiation with respect to

* Received November 15, 1937.

u^1 and u^2 respectively. The normal vector of S is defined as the vector (2) divided by its length and will be denoted by $\boldsymbol{x}_3$. Thus one has

$$(3)\qquad (\boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{x}_3) > 0 \quad \text{and} \quad \boldsymbol{x}_3(\boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{x}_3) = (\boldsymbol{x}_1 \times \boldsymbol{x}_2).$$

It is well known that the area of a region T of S may be represented in the form

$$(4)\qquad \iint_T (\boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{x}_3)\,du^1du^2 = \iint_T d\sigma.$$

Similarly, if

$$(5)\qquad \boldsymbol{x} = \boldsymbol{x}_3(u^1, u^2)$$

represents, in case of a surface S of class C_2, the spherical image of S, one obtains for the area of the spherical image of a region T of S

$$(6)\qquad \int_T (\boldsymbol{x}_{31}\boldsymbol{x}_{32}\boldsymbol{x}_3)\,du^1du^2,$$

where the usual conventions have to be used if the correspondence between T and its image on the sphere is not one-to-one. One may define the Gaussian curvature K of S at a point P of S as the limit of the quotient of the area of T and of its spherical image, when T is a variable part of S of simple form (e. g. a u^i-square) which has P as limit. One finds, from (4) and (6)

$$(7)\qquad (\boldsymbol{x}_{31}\boldsymbol{x}_{32}\boldsymbol{x}_3) = K(\boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{x}_3).$$

This well-known formula may be taken immediately from the usual definition of K by means of the Weingarten differentiation formulae.

2. Let an arc R of class C_2 on a surface S of class C_2 be represented in the form

$$(8)\qquad u^i = u^i(s),$$

where s is the arc length of R. On substituting (8) in (1) and differentiating, one obtains

$$(9)\qquad \dot{\boldsymbol{x}} = \boldsymbol{x}_i\,\dot{u}^i,$$

where the $\dot{}$ denotes differentiation with respect to s and the tangent vector $\dot{\boldsymbol{x}}$ of R has length 1 since the arc length is parameter on R.

It is clear that there is a one-to-one correspondence between vectors in the u^i-plane which are attached at a certain point and vectors tangent to S at the

corresponding point of S and that this correspondence is continuous both ways. For instance to (9) there corresponds the vector $\dot{u}^i$ attached at the point (8).

A system

$$(10) \qquad \boldsymbol{v} = \boldsymbol{v}(s)$$

of vectors which is of class C_1 and is tangent to S at the point (8) of S is called *parallel along R* if

$$(11) \qquad \dot{\boldsymbol{v}} = \dot{\boldsymbol{v}}(s) \text{ is normal to } S \text{ at } (8).$$

On placing $\boldsymbol{v} = v^i \boldsymbol{x}_i$ and $\dot{\boldsymbol{v}} \cdot \boldsymbol{x}_i = 0$ one obtains linear differential equations for the $v^i(s)$, so that a system (10) exists and is uniquely determined if the $v^i(s)$ are given for one value of s. Since $\dot{\boldsymbol{v}}$ is perpendicular to $\boldsymbol{v}$, the length of $\boldsymbol{v}$ is independent of s. *If* (10) *is parallel along R, then so is any vector obtained from* (10) *by rotation of* (10) *in the tangent plane of S at* (8) *over an angle which is independent of s.* For if (10) is parallel, so is $\boldsymbol{w} = \boldsymbol{x}_3 \times \boldsymbol{v}$ which is obtained from $\boldsymbol{v}$ by a rotation over the angle $\frac{1}{2}\pi$; in fact, $\dot{\boldsymbol{w}} = \dot{\boldsymbol{x}}_3 \times \boldsymbol{v} + \boldsymbol{x}_3 \times \dot{\boldsymbol{v}}$ is normal to S at (8), since $\dot{\boldsymbol{x}}_3$ is tangent and $\dot{\boldsymbol{v}}$ is normal. Furthermore if any two vectors are parallel along R, so are their linear combinations with constant coefficients.

It follows that, if ϑ is the angle from (10) to (9), then $\dot{\vartheta}$ is independent of the particular system (10) chosen. The derivative $\dot{\vartheta}$ is called the *geodesic* curvature of R and will be denoted by κ. One finds, if $\boldsymbol{v}$ has length 1,

$$(12) \qquad \sin\vartheta = (\boldsymbol{v}\dot{\boldsymbol{x}}\boldsymbol{x}_3), \qquad \cos\vartheta = \boldsymbol{v}\cdot\dot{\boldsymbol{x}},$$

hence

$$\dot{\vartheta}(\boldsymbol{v}\cdot\dot{\boldsymbol{x}}) = (\dot{\boldsymbol{v}}\dot{\boldsymbol{x}}\boldsymbol{x}_3) + (\boldsymbol{v}\ddot{\boldsymbol{x}}\boldsymbol{x}_3) + (\boldsymbol{v}\dot{\boldsymbol{x}}\dot{\boldsymbol{x}}_3).$$

Here the first and third triple products are 0, since $\dot{\boldsymbol{v}}$ is normal and $\dot{\boldsymbol{x}}_3$ is tangent to S. On choosing $\boldsymbol{v}$ to be $\dot{\boldsymbol{x}}$ at the point of R under consideration, one finds $\boldsymbol{v}\cdot\dot{\boldsymbol{x}} = 1$, hence

$$(13) \qquad \kappa = \dot{\vartheta} = (\dot{\boldsymbol{x}}\ddot{\boldsymbol{x}}\boldsymbol{x}_3).$$

3. Let a vector $\boldsymbol{w} = \boldsymbol{w}(s)$ be tangent to S at the point (8), let $\boldsymbol{w}$ be of class C_1 and of length 1 and let the angles from $\boldsymbol{w}$ to $\boldsymbol{x}$ and to $\boldsymbol{v}$ be denoted by ω and ϕ. Then one has

$$(14) \qquad \omega = \vartheta + \phi,$$

if care is taken that all three angles are continuous and that (14) holds exactly (and not only modulo 2π) at some point of R. Furthermore

$$(15) \qquad \sin\phi = (\boldsymbol{w}\boldsymbol{v}\boldsymbol{x}_3), \qquad \cos\phi = \boldsymbol{w}\cdot\boldsymbol{v},$$

and so, in the same way as (13) from (12),

$$\dot{\phi} = (\dot{\boldsymbol{w}}\boldsymbol{w}\boldsymbol{x}_3). \tag{16}$$

Now let an arc R be divided by a finite number of vertices P_k into arcs of class C_2. Denote by α_k the angle from $\dot{\boldsymbol{x}}(s_k - 0)$ to $\dot{\boldsymbol{x}}(s_k + 0)$, where $-\pi \leqq \alpha_k \leqq \pi$ and s_k is the value of s at P_k. If $\alpha_k = \pm\pi$ choose the signature of α_k in such a way that at every P_k the angle subtended by the region to the right of R is $\alpha_k + \pi$.

Let $\boldsymbol{w}$ be continuous at P_k and of class C_1 on the remainder of R. Clearly a parallel system $\boldsymbol{v}$ may be defined along R having the same properties. If the convention is used that

$$\vartheta(s_k + 0) - \vartheta(s_k - 0) = \alpha_k, \qquad \omega(s_k + 0) - \omega(s_k - 0) = \alpha_k, \tag{17}$$

while ϕ is continuous for every s, then (14) remains true for the arc R. If $\Delta\omega, \Delta\vartheta, \Delta\phi$ denotes the increase of ω, ϑ, ϕ, while s increases from its initial value to its final value, one has, by (14) and (17),

$$\Delta\omega = \Delta\vartheta + \Delta\phi \tag{18}$$

and one obtains from (13) and (17) and from (16)

$$\Delta\vartheta = \int_R \kappa\, ds + \Sigma_k\alpha_k \quad \text{and} \quad \Delta\phi = \int_R (\dot{\boldsymbol{w}}\boldsymbol{w}\boldsymbol{x}_3)\, ds, \tag{19}$$

hence, from (18),

$$\Delta\omega = \int_R \kappa\, ds + \int_R (\dot{\boldsymbol{w}}\boldsymbol{w}\boldsymbol{x}_3)\, ds + \Sigma_k\alpha_k. \tag{20}$$

4. The meaning of (20) is not lost if R is a simple closed curve and the parameter s begins and ends at a point of R where the second derivative exists. It will be assumed for the present that R is a simple closed curve which is the positively oriented boundary of a compact portion T of S. In that case it is easy to define a vector field $\boldsymbol{w}$ which is of class C_1 on T, hence continuous on R and of class C_1 on each arc of class C_2 of R. One can for instance transfer a suitable field from the u^i-plane to S. It will be shown that in the case considered

$$\Delta\omega = 2\pi \tag{21}$$

First, $\Delta\omega = \Delta\lambda$, where λ is the angle in the u^i-plane, from the vector w^i which corresponds to $\boldsymbol{w}$ to the vector $\dot{u}^i$ which corresponds to $\dot{\boldsymbol{x}}$. This is clear,

since both $\Delta\omega$ and $\Delta\lambda$ are integer multiples of 2π, while, for increasing s, the angles ω and λ both pass at the same time and in the same direction through a multiple of π.

Next,[1] $\Delta\lambda = \Delta\mu$, where μ is the inclination of u^i, i. e. the angle from the vector $(1, 0)$ to the vector (u^1, u^2). Note that T, which is compact by assumption, is represented in the u^i-plane by the simple closed curve R together with its interior. Now on this set the inclination of w^i may be determined as a continuous function.[2] Hence w^i may be changed into the constant vector field $(1, 0)$ by a continuous rotation. This proves $\Delta\lambda = \Delta\mu$, since both $\Delta\lambda$ and $\Delta\mu$ are integral multiples of 2π.

In order to prove (21) it remains to show that $\Delta\mu = 2\pi$. Now this is the "Umlaufsatz" of which simple proofs may be found in the literature.[3]

As a consequence of (21), (20) takes the form

$$\int_R \kappa\, ds + \int_R (\dot{\boldsymbol{w}}\boldsymbol{w}\boldsymbol{x}_3)\, ds = 2\pi - \Sigma_k \alpha_k. \tag{22}$$

5. Let it be assumed for a moment that S is of class C_3, so that $\boldsymbol{w}$ may be chosen of class C_2. Then the theorem of Green may be applied to the second integral in (22) as follows,

$$\begin{aligned} \int_R (\dot{\boldsymbol{w}}\boldsymbol{w}\boldsymbol{x}_3)\, ds &= \int_R [(\boldsymbol{w}_1\boldsymbol{w}\boldsymbol{x}_3)\, du^1 + (\boldsymbol{w}_2\boldsymbol{w}\boldsymbol{x}_3)\, du^2] \\ &= \iint_T [(\boldsymbol{w}_2\boldsymbol{v}\boldsymbol{x}_3)_1 - (\boldsymbol{w}_1\boldsymbol{w}\boldsymbol{x}_3)_2]\, du^1 du^2. \end{aligned} \tag{23}$$

Now one has for the integrand of the last integral,

$$\begin{aligned} (\boldsymbol{w}_2\boldsymbol{w}\boldsymbol{x}_3)_1 - (\boldsymbol{w}_1\boldsymbol{w}\boldsymbol{x}_3)_2 &= (\boldsymbol{w}_{21}\boldsymbol{w}\boldsymbol{x}_3) \\ &+ (\boldsymbol{w}_2\boldsymbol{w}_1\boldsymbol{x}_3) + (\boldsymbol{w}_2\boldsymbol{w}\boldsymbol{x}_{31}) - (\boldsymbol{w}_{12}\boldsymbol{w}\boldsymbol{x}_3) - (\boldsymbol{w}_1\boldsymbol{w}_2\boldsymbol{x}_3) - (\boldsymbol{w}_1\boldsymbol{w}\boldsymbol{x}_{32}). \end{aligned} \tag{24}$$

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[1] It is clear that this step in the argument could be avoided by choosing w in a suitable way to begin with.

[2] It is possible to subdivide T into arbitrarily small polygons with edges of class C_2. One may for instance use line segments in the u^i-plane. If μ could not be defined as a continuous function on T, the variation of μ along the boundary of at least one of these polygons would be $2\pi n \neq 0$. But this is clearly impossible if the polygons are sufficiently small.

[3] H. Hopf, *Compositio Mathematica*, vol. 2 (1935), p. 50 and pp. 53-55, where further references are given; also E. R. van Kampen, *Compositio Mathematica*, vol. 4 (1937), p. 272.

Here the first and fourth terms on the right cancel, while the second and fifth are both zero, since they contain only vectors perpendicular to $\boldsymbol{w}$. Hence

$$\int_R (\dot{\boldsymbol{w}}\boldsymbol{w}\boldsymbol{x}_3)\,ds = \int_T\!\!\int [(\boldsymbol{w}_2\boldsymbol{w}\boldsymbol{x}_{31}) - (\boldsymbol{w}_1\boldsymbol{w}\boldsymbol{x}_{32})]\,du^1du^2. \tag{25}$$

It remains to evaluate the integrand of the last integral. Now $\boldsymbol{w}$ and $\boldsymbol{x}_{31}$ are tangent to S, so that in $(\boldsymbol{w}_2\boldsymbol{w}\boldsymbol{x}_{31})$, $\boldsymbol{w}_2$ may be replaced by its normal component $(\boldsymbol{w}_2\cdot\boldsymbol{x}_3)\boldsymbol{x}_3$ which is equal to $-(\boldsymbol{w}\cdot\boldsymbol{x}_{32})\boldsymbol{x}_3$, since $\boldsymbol{w}\cdot\boldsymbol{x}_3 = 0$. Thus one finds for the integrand on the right of (25),

$$(\boldsymbol{x}_3\boldsymbol{w}\boldsymbol{x}_{32})(\boldsymbol{w}\cdot\boldsymbol{x}_{31}) - (\boldsymbol{x}_3\boldsymbol{w}\boldsymbol{x}_{31})(\boldsymbol{w}\cdot\boldsymbol{x}_{32}). \tag{26}$$

Applying the identity $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) = (\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}$ in two different ways to $\boldsymbol{a}\cdot((\boldsymbol{b}\times\boldsymbol{c})\times(\boldsymbol{d}\times\boldsymbol{e}))$ one obtains

$$\boldsymbol{a}\cdot\boldsymbol{b}(\boldsymbol{cde}) - \boldsymbol{a}\cdot\boldsymbol{c}(\boldsymbol{deb}) + \boldsymbol{a}\cdot\boldsymbol{d}(\boldsymbol{ebc}) - \boldsymbol{a}\cdot\boldsymbol{e}(\boldsymbol{bcd}) = 0.$$

Substituting here $\boldsymbol{w}, \boldsymbol{w}, \boldsymbol{x}_3, \boldsymbol{x}_{31}, \boldsymbol{x}_{32}$ instead of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}$, one finds that (26) equals $(\boldsymbol{x}_{31}\boldsymbol{x}_{32}\boldsymbol{x}_3)$. Thus (26) is $K(\boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{x}_3)$ by (7), and (4) shows that (25) takes the form

$$\int_R (\dot{\boldsymbol{w}}\boldsymbol{w}\boldsymbol{x}_3)\,ds = \int_T\!\!\int K\,d\sigma, \tag{27}$$

so that the theorem of Gauss-Bonnet (in the simplest case) is obtained from (22) in the form:

$$\int_R \kappa\,ds + \int_T\!\!\int K\,d\sigma = 2\pi - \Sigma_k\alpha_k. \tag{28}$$

The formula

$$(\boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{x}_3)K = (\boldsymbol{w}_2\boldsymbol{w}\boldsymbol{x}_3)_1 - (\boldsymbol{w}_1\boldsymbol{w}\boldsymbol{x}_3)_2,$$

which is implicit in (23) and (27) goes over into well-known formulae, if one chooses $\boldsymbol{w}$ to be parallel either to $\boldsymbol{x}_1$ or to $\boldsymbol{x}_2$. It is obvious that the identity of (26) and $(\boldsymbol{x}_{31}\boldsymbol{x}_{32}\boldsymbol{x}_3)$ may be obtained from the rule for $\sin(\alpha - \beta)$ where α and β are the angles from $\boldsymbol{x}_{31}$ and $\boldsymbol{x}_{32}$ to $\boldsymbol{w}$.

6. Now it will be shown that (25) holds for a vector field $\boldsymbol{w}$ of class C_1. If $a = a(u^1, u^2)$ and $b = b(u^1, u^2)$ are scalar functions on T of class C_1, then

$$\int a\dot{b}\,ds = \int_R (ab_1\,du^1 + ab_2\,du^2) = \int_T\!\!\int (a_1b_2 - a_2b_1)\,du^1du^2. \tag{29}$$

This is a consequence of the theorem of Green, since b, b_1 and b_2 may be approximated uniformly on T by a polynomial in u^1 and u^2 and its two partial derivatives. On replacing b in (29) by the different components of $\boldsymbol{w}$, one may verify that

$$\int_R (\boldsymbol{w} \times \boldsymbol{x}_3) \cdot \dot{\boldsymbol{w}} ds = \iint_T [(\boldsymbol{w} \times \boldsymbol{x}_3)_1 \cdot \boldsymbol{w}_2 - (\boldsymbol{w} \times \boldsymbol{x}_3)_2 \cdot \boldsymbol{w}_1] du^1 du^2,$$

which identity is equivalent to (25), since $(\boldsymbol{w}_1 \boldsymbol{w}_2 \boldsymbol{x}_3) = 0$.

Since a vector field $\boldsymbol{w}$ of class C_1 always exists on T, provided that S is of class C_2, it follows that (28) holds if S is of class C_2, while R consists of arcs of class C_2.

Since, as is well known, on a surface of class C_2 the geodesic curvature of an arc R depends on the first fundamental form of S only, and since (28) holds for every T, it follows that the Gaussian curvature of a surface S of class C_2 does not depend on the second fundamental form of S. Hence the *Theorema Egregium* holds for surfaces of class C_2.

7. It is, of course, not possible to represent every surface S by one representation (1). However, any surface S may be defined by means of a finite or enumerable collection of representations (1), with the understanding that whenever a part of S is defined by two representations (1), there exists an orientation preserving continuous one-to-one correspondence between the two sets of parameters involved. The parameter representations (1) and the above mentioned correspondence are all supposed to be of the same class C_n, which is also the class of the surface S. Now, *any compact subset T of S, which has as boundary R a graph consisting of arcs of class C_n, may be subdivided by additional arcs of class C_n into a finite number of regions T_m, $m = 1, 2, \cdots$, such that the boundary R_m of T_m is a simple closed curve and that T_m is defined by means of only one of the representations* (1) *of S.* The proof may be sketched as follows. First, as a consequence of the compactness of T, this set is contained in the part of S defined by means of a finite number of representations (1). Next, the part of T defined by a fixed representation (1) is a subset of the corresponding u^i-plane of which the boundary consists of certain arcs of class C_n. The part of T which is not defined by any other representation (1) and not yet subdivided in the desired way is bounded in this u^i-plane. Clearly a somewhat larger part of T may be subdivided in the desired way, for instance by means of line segments. On repeating this for the finite number of representations (1) which are used to define T, one clearly obtains a proof of the above statement.

If the numbers of vertices, edges, regions T_m, which occur in the above subdivision is a_0, a_1, a_2, one terms $C = a_0 - a_1 + a_2$ the *characteristic of* T. In case $n = 2$, the independence of C from the particular subdivision is an automatic consequence of the Gauss-Bonnet theorem.

8. Now suppose that a region T on a surface S of class C_2 has a boundary R consisting of arcs of class C_2 and that $a_0, a_1, a_2, T_m, R_m, m = 1, \cdots, a_2$, have the meaning of **7**. Denote by β_{mk} the *interior* angles of T_m, so that, since (28) holds for each T_m,

$$\int_{R_m} \kappa \, ds + \int\int_{T_m} K \, d\sigma = 2\pi - \Sigma_k(\pi - \beta_{mk}). \tag{30}$$

On taking the sum of (30) for all m one obtains

$$\int_{R} \kappa \, ds + \int\int_{T} K \, d\sigma = 2\pi a_2 - \Sigma_m \Sigma_k(\pi - \beta_{mk}). \tag{31}$$

Let β_k, $k = 1, 2, \cdots$, denote all interior angles of the complement of T on S, so that $\alpha_k = \beta_k - \pi$ are the discontinuities in the tangent vector of R if one follows all boundaries of the complement of T in their negative direction. The α_k will be called the oriented angles of R. The right side of (31) is

$$2\pi a_2 - \Sigma_k \alpha_k - \Sigma_m \Sigma_k(\pi - \beta_{mk}) - \Sigma_k(\pi - \beta_k),$$

or

$$2\pi a_2 - \Sigma_k \alpha_k + (\Sigma_m \Sigma_k \beta_{mk} + \Sigma_k \beta_k) - (\Sigma_m \Sigma_k \pi + \Sigma_k \pi). \tag{32}$$

Here the third term represents the sum of all angles subtended at all vertices of the subdivisions of T used, hence it is equal to $2\pi a_0$. The fourth term contains the number π each time a region T_m (or the complement of T) adjoins a vertex. Hence it also contains the number π each time an edge adjoins a vertex. Thus it is equal to $2\pi a_1$. On using the definition of the characteristic C of T, one sees that (31) goes over into

$$\int_{R} \kappa \, ds + \int\int_{T} K \, d\sigma = 2\pi C - \Sigma_k \alpha_k. \tag{33}$$

Thus one may formulate the general

THEOREM OF GAUSS-BONNET. *If K and $d\sigma$ are Gaussian curvature and area element of a surface S of class C_2, T is a compact region of S with char-*

acteristic C and of which the boundary R consists of a finite number of arcs of class C_2; if furthermore κ is the geodesic curvature and ds the element of length of the oriented boundary R of T; if finally the oriented angles of R are denoted by α_k, then (33) *holds.*

It may be remarked that the existence of a field $\boldsymbol{w}$ of constant length and of class C_2 may be proved on T, whenever the boundary R of T is not the empty set. For such a field the considerations of this section prove the formula $\Delta\omega = 2\pi C$, where ω is the angle from $\boldsymbol{w}$ to the tangent vector of R, and $\Delta\omega$ is the variation of ω along R. This formula may be considered as the generalisation of the "Umlaufsatz" for curved surfaces. The proof may of course be based on less regularity assumptions.

10. Stokes Theorem. The result of **7** may be used to give a simple derivation [4] of Stokes' theorem for a vector field and surface with boundary all of class C_1.

Let S, T and R have the properties of **7** in case $n = 1$, and let $\boldsymbol{v}$ be a vector field of class C_1 defined on a region U which has S in its interior.[5] Since Stokes' theorem clearly follows for T if it is proved for the regions T_m defined in **7**, it may be assumed that S is defined in terms of a single representation (1) and that T is a compact region with a simple closed curve R as boundary.

Let $\boldsymbol{\partial}$ represent a vectorial differential operator of which the action extends only to the vector field $\boldsymbol{v}$. Thus one has, for instance,

$$\boldsymbol{\partial} \cdot \boldsymbol{v} = \operatorname{div} \boldsymbol{v}, \qquad \boldsymbol{\partial} \times \boldsymbol{v} = \operatorname{rot} \boldsymbol{v}, \tag{34}$$

and on the surface S,

$$(\boldsymbol{x}_1 \cdot \boldsymbol{\partial})\boldsymbol{v} = \boldsymbol{v}_1, \qquad (\boldsymbol{x}_2 \cdot \boldsymbol{\partial})\boldsymbol{v} = \boldsymbol{v}_2. \tag{35}$$

Applying the well-known identity

$$(\boldsymbol{a} \times \boldsymbol{b}) \cdot (\boldsymbol{c} \times \boldsymbol{d}) = (\boldsymbol{a} \cdot \boldsymbol{c})(\boldsymbol{b} \cdot \boldsymbol{d}) - (\boldsymbol{a} \cdot \boldsymbol{d})(\boldsymbol{b} \cdot \boldsymbol{c})$$

to $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{\partial}, \boldsymbol{v}$ instead of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$, one obtains on S,

$$(\boldsymbol{x}_1 \times \boldsymbol{x}_2) \cdot (\boldsymbol{\partial} \times \boldsymbol{v}) = (\boldsymbol{x}_1 \cdot \boldsymbol{\partial})(\boldsymbol{v} \cdot \boldsymbol{x}_2) - (\boldsymbol{x}_2 \cdot \boldsymbol{\partial})(\boldsymbol{v} \cdot \boldsymbol{x}_1),$$

[4] A similar proof is given in McShane's translation of Courant's *Differential und Integral Rechnung.*

[5] It would be sufficient for the boundary R of T to consist of rectifiable curves, since in that case Green may be applied also.

hence by (34) and (35),

$$(\boldsymbol{x}_1\boldsymbol{x}_2 \operatorname{rot} \boldsymbol{v}) = \boldsymbol{v}_1 \cdot \boldsymbol{x}_2 - \boldsymbol{v}_2 \cdot \boldsymbol{x}_1,$$

and by (3),

$$(\operatorname{rot} \boldsymbol{v} \cdot \boldsymbol{x}_3)(\boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{x}_3) = \boldsymbol{v}_1 \cdot \boldsymbol{x}_2 - \boldsymbol{v}_2 \cdot \boldsymbol{x}_1. \tag{36}$$

The validity of this symbolic computation is obvious.

On applying Green's theorem in the form (29) to the different components of $\boldsymbol{v}$ and $\dot{\boldsymbol{x}}$ one finds

$$\int_R \boldsymbol{v} \cdot \dot{\boldsymbol{x}}\, ds = \int_T\!\!\int (\boldsymbol{v}_1 \cdot \boldsymbol{x}_2 - \boldsymbol{v}_2 \cdot \boldsymbol{x}_1)\, du^1 du^2$$

or, by (36) and (4)

$$\int_R \boldsymbol{v} \cdot \dot{\boldsymbol{x}}\, ds = \int_T\!\!\int \operatorname{rot} \boldsymbol{v} \cdot \boldsymbol{x}_3\, d\sigma. \tag{37}$$

On defining $\dot{\boldsymbol{x}}\, ds = d\boldsymbol{s}$ and $\boldsymbol{x}_3 d\sigma = d\boldsymbol{\sigma}$ to be the vectorial elements of length and area of R and T one obtains the

THEOREM OF STOKES. *If T is a compact region of a surface S of class C_1 and the boundary of T consists of a finite number of arcs of class C_1, if furthermore $d\boldsymbol{s}$ and $d\boldsymbol{\sigma}$ are the vectorial elements of length and area of R and T and $\boldsymbol{v}$ is a vector field of class C_1 in a region U which contains S, then*

$$\int_R \boldsymbol{v} \cdot d\boldsymbol{s} = \int_T\!\!\int \operatorname{rot} \boldsymbol{v} \cdot d\boldsymbol{\sigma}. \tag{38}$$

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HOMOMORPHISM OF RINGS AND FIELDS OF POINT SETS.*

By Morris Kline.

1. Introduction. Though systems of sets of points such as covering systems, systems of sets defining a space, Borel, and analytic sets have been extensively studied, little attention has been paid to questions involving properties analogous to group properties of such or more general systems. Recently [1] M. H. Stone has studied the abstract algebraic properties of Boolean rings and their connection with Boolean algebras, and his work has significance for systems of point sets which are interpretations of Boolean algebras. The rings and fields of point sets studied in this paper do constitute interpretations of generalized Boolean algebras, and the complete fields, a term to be defined shortly, here studied, constitute an interpretation of the Boolean algebras. Hence some of the group properties of rings and fields of point sets are already known. Yet, because the point set systems are special cases, we are permitted to obtain results for them thus far not obtained for the more general systems and perhaps of importance only for point sets.

The work of this paper has bearing on existing material in that many systems of point sets, such as Borel sets and the system of all subsets of a given set, form complete fields; [2] in that a ring of point sets has an immediate interpretation as a ring of functions and conversely; [3] and in that the notion of homomorphism between set systems is useful in topological problems.[4] Also, the possibilities of combining point set and group properties as in the study of topological groups are interesting.

2. Rings and fields of sets. This part will deal with the simpler algebraic properties of rings and fields of point sets. Not all the results in this part are new; in a few instances, as indicated below, the conclusions are implied by work on Boolean rings. Nevertheless, a few proofs are given of

* Received February 10, 1937; revised August 31, 1937.

[1] M. H. Stone, "The theory of representations for Boolean algebras," *Transactions of the American Mathematical Society*, vol. 40 (1936), pp. 37-111.

[2] C. Kuratowski, *Topologie* I, p. 22.

[3] W. Sierpiński, "Sur les anneaux de fonctions," *Fundamenta Mathematicae*, vol. 18 (1932), p. 6.

[4] See, for example, E. Čech, "Théorie générale de l'homologie dans un espace quelconque," *Fundamenta Mathematicae*, vol. 19 (1932), pp. 149-183.

old results because they can be obtained so much more easily for systems of point sets.

DEFINITIONS. *The sum of two systems A and B of point sets, indicated by $A + B$, shall mean the system of sets obtained by adding each set of A to each set of B, as point sets; likewise for the product, indicated by $A \times B$, and the difference, $A - B$, of two systems. A single set is sometimes regarded as a system in the use of these operations.*

A ring of sets is a system of sets to which the sum and intersection of any two sets of the system belong.[5] By a subring we shall understand any subcollection of sets of the ring which also form a ring.

If R is a ring, by the homomorph of R we shall mean a system of sets R' satisfying the following conditions:[6]

1) To each set of R there shall correspond one and only one set of R';
2) Each set of R' shall correspond to at least one set of R;
3) If $A + B = C$ or $A \cdot B = D$, where A and B are any sets of R, corresponding relations shall hold among the corresponding sets of R'. The relation in R' corresponding to one in R will be the same as the latter unless otherwise specified.

If, instead of condition 2), we have that to each set of R' there corresponds one and only one set of R we shall say that R' is the isomorph of R.

By a residue class of a ring R with respect to a subring S we mean the system $A + S$, where A belongs to R. An ideal is a subring such that if set B belongs to the subring and C is any set of the ring, then $B \cdot C$ belongs to the subring.

The very definition of homomorphism gives the following occasionally useful two theorems.

THEOREM I. *The homomorph of a ring is a ring.*

THEOREM II. *All the residue classes of a ring R with respect to an ideal S form a system of residue classes which is the homomorph of R if we let sum and intersection in R correspond to sum and product, respectively, of the residue classes.*[7]

THEOREM III. *If R is a ring containing a null set,*[8] *the system of residue*

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[5] F. Hausdorff, *Mengenlehre* (1935), p. 77.

[6] B. L. van der Waerden, *Moderne Algebra*, vol. 1, p. 44.

[7] Cf. van der Waerden, *loc. cit.*, p. 35.

[8] The term null set is here used to mean the empty set or a zero set.

classes determined by any ideal S of R is the isomorph of R under the correspondence of operations of the previous theorem.

Proof. If A and B belong to R, then $A + S = B + S$ if and only if $A = B$. For, since R contains a null set and S is an ideal, S contains a null set. Suppose $A + S = B + S$. Then $A + 0 = B + P$ where P belongs to S. Hence $A \supset B$. Likewise $B + 0 = A + Q$ where Q belongs to S. Hence $B \supset A$, and therefore $A = B$.

It is now obvious that if we let A correspond to $A + S$ the correspondence will, in view of the previous theorem, determine an isomorphism.

DEFINITION. *By a domain of integrity we shall mean a ring in which the intersection of any two non-zero elements is a non-zero element.*

THEOREM IV. *A necessary and sufficient condition that the residue classes with respect to an ideal S of a ring R containing a null set form a domain of integrity*[9] *is that R be a domain of integrity.*

Proof. If A and B belong to R and $A \cdot B = 0$, then the product of the corresponding residue classes would have to give S, else the isomorphism just proven to exist would not hold. Likewise for the converse.

DEFINITION. *If the system R' is the homomorph of the system R, then the sets of R which correspond to the set A' of R' are said to form the class of R corresponding to A'.*[10]

THEOREM V. *If the set system R' is the homomorph of the set system R and R' contains a null set, the class Σ of R corresponding to this null set, $0'$, is an ideal. The other classes are domains of integrity.*[11]

Proof. That Σ is an ideal follows as in van der Waerden. If K_A denotes the system of sets of R corresponding to $A' \neq 0'$ of R', then certainly if A and B belong to K_A, $A + B$ and $A \cdot B$ belong. Moreover, under any homomorphism of two rings, if R contains a null set it must correspond to $0'$ of R'; for, if M be any set of R, since $M \cdot 0 = 0$, $M' \cdot A' = A'$, where 0 corresponds to A'. But M' can be any set of R'; hence A' is included in every set of R' including the null set $0'$. Hence $A' = 0'$. In view of this fact, if A and B belong to K_A, $A \cdot B \neq 0$, else $A' \cdot A' = 0'$. Hence K_A is a domain of integrity.

DEFINITIONS. *A field of sets is a system of sets to which the sum and*

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[9] The zero residue class is S, itself.

[10] Cf. van der Waerden, *loc. cit.*, p. 32.

[11] Cf. van der Waerden, *loc. cit.*, p. 56.

difference of any two sets of the system belong.[12] *A subfield shall mean any subcollection of sets of the field which is itself a field. By a complete field we shall mean a field to which the sum of all the sets of the field belongs.*

If A and B are any two sets, $A \cdot B = A - [(A + B) - B]$; hence a field is a ring. Hence by the homcmorph of a field we shall understand a system of sets satisfying the conditions of a homomorph of a ring. Similarly for isomorphism.

The following result is not new but can be obtained at once for systems of point sets.[13]

THEOREM VI. *Every field of finite order (i. e., field containing a finite number of sets) consists exclusively of sets formed by the process of addition applied to a subcollection of mutually exclusive sets of the field.*

Let the mutually exclusive sets be lettered $B_1, B_2, \cdots, B_n$. Then every set of the field is easily shown to be expressible in the form $A = \sum_{j=1}^{n} b_j B_j$, where each b_j is 0 or 1. From this statement we have

COROLLARY I. *Every field of finite order contains* 2^n *sets.*[14]

COROLLARY II. *The order of a subfield is a divisor of the order of the field.*

THEOREM VII. *Let the system F' of sets be the homomorph of the field F. Then there exists a field F'' of sets formed from the sets of F', which is the homomorph of F. Moreover, the correspondence between F and F'' holds under the operation of difference.*

Proof. Let the null set of F correspond to the set D' of F'. If A is any set of F, since $A \cdot 0 = 0$, $A' \cdot D' = D'$; and D' is therefore contained in every set of F'.

Let the system F'' consist of all sets of the form $A' - D'$, where A' is any set of F', and let A of F correspond to $A' - D'$ of F''. If the correspondence between F and F'' holds under sum, intersection, and difference it will follow that F'' is a field.

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[12] Hausdorff, *loc. cit.*, p. 78. Hausdorff uses difference only when the subtrahend is a subset of the minuend. We do not. This distinction is unimportant for fields because, if A and B are any two sets, $A - B = (A + B) - B$ and hence under either use of the term the same sets belong to a field.

[13] Cf. theorems 4 and 12 in the paper of Stone's referred to above.

[14] This result is obtained by B. A. Bernstein, "On finite Boolean algebras," *American Journal of Mathematics*, vol. 57 (1935), p. 742.

That the correspondence holds under sum and intersection is immediate. To show that it holds under difference we have but to employ the fact that $A - B = C$ implies $A = A \cdot B + C$ and $(A \cdot B) \cdot C = 0$.

We should notice that if F' contains a null set, $0'$, then $D' = 0'$ and $F' = F''$. Moreover, if F is a complete field with S as the sum of its sets,[15] then if S' of F' is the correspondent of S, S' must be the sum of the sets of F'. This follows because if A is any set of F we have $S \cdot A = A$; then $S' \cdot A' = A'$. These facts give the

COROLLARY. *If the set system F' is the homomorph of the field (complete field) F, where F' contains a null set, then F' is a field (complete field). Moreover, the correspondence between F and F' holds under the operation of difference.*[16]

It is easy to prove that

THEOREM VIII. *The homomorph F' of a complete field F, under a correspondence of sum and intersection in F with intersection and sum, respectively, in F', is a complete field provided F' contains a null set.*

THEOREM IX. *If F' is a set system which is the homomorph of the field F and if F' contains a null set, then the class Σ of F which corresponds to $0'$ is a field, and the sets of any other class are expressible in terms of the sum and difference of the sets of Σ and some one member of that class.*[17]

Proof. If A and B of F correspond to $0'$ of F', then $A + B$ and $A - B$ correspond to $0'$ because theorem VII assures us that the correspondence holds under difference.

Let A of F correspond to $A' \neq 0'$ of F'. Suppose B of F corresponds to A'. We shall show that B is expressible as the sum and difference of A and the sets of Σ. Let $A - B = X$. Since $A - B$ corresponds to $A' - A' = 0'$, $A - B$ belongs to Σ. Likewise $B - A = Y$ belongs to Σ. Finally, $B = A - X + Y$.

COROLLARY. *If Σ consists of the null set only, F is isomorphic to F'.*

THEOREM X. *If the set system F' is the homomorph of the field F; if F' contains a null set; and if the classes of F each contain a finite number of sets, then any class of F is a residue class with respect to Σ where Σ is the class of F which corresponds to $0'$.*

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[15] We refer to S later as the unit element of F.

[16] This last statement is contained in theorem 42 of the paper of Stone's referred to above.

[17] Cf. van der Waerden, *loc. cit.*, p. 56.

Proof. Suppose K_A is the class of sets which correspond to A' of F'. Since K_A contains a finite number of sets, the intersection of all these sets, $\bar{A}$ say, is a set of K_A since K_A is a ring by theorem V. As in the preceding theorem, if A belongs to K_A, then $A - \bar{A}$ belongs to Σ. Then the class K_A is the residue class $\bar{A} + \Sigma$, for $A = \bar{A} + (A - \bar{A})$ and this last is a set of $\bar{A} + \Sigma$.

Suppose the points of space T' are the transforms under the single-valued function, f, of the points of space T. If A is a set in T, let us understand the corresponding set in T' to mean the set of points of T' which correspond to the points of A. We then have the

THEOREM XI. *Let the points of space T' be single-valued transforms of the points of space T. Let F be a field, or complete field, of sets in T such that, if A and B are any sets of F for which $A \cdot B = 0$ then for the corresponding sets A' and B' in T', $A' \cdot B' = 0$.* The corresponding sets in T' form a field, or complete field, isomorphic to F.

Proof. Under any single-valued transformation we have that if $A + B = C$, where A and B are here sets of F, then $A' + B' = C'$.

Suppose $A \cdot B = D$. Again we have at once that $D' \subset A' \cdot B'$ and we show first that $D' \supset A' \cdot B'$. Suppose p' belongs to A' and to B'. Then some correspondent of p', p say, is in A, and some correspondent, q say, is in B. If $p = q$, p' is in D'. If $p \neq q$ but both p and q belong to A and B, then p' is in D'. If $p \neq q$, and q does not belong to A say, then we shall have $A \cdot (B - A) = 0$, but $A' \cdot (B - A)' \neq 0$, contrary to hypothesis. Hence $D' = A' \cdot B'$.

We have, so far, that the system F' of sets in T' is the homomorph of F. The corollary to theorem VII gives us the fact that F' is a field (complete if F is complete). We note further that the only set of F which can correspond to the null set of F' is the null set of F. Hence, by the corollary to theorem IX, F' is the isomorph of F.

3. The correspondence of sums and limits of sequences of sets.

It is possible to have two fields isomorphic to each other and for the first field to contain the sum of a countable number of its sets without the second field containing the sum of the corresponding sets. The following is an example of such a situation. Let the field F consist of a system of concentric circles of radii $1/2, 3/4, 7/8, \cdots$, a circle of radius 2, and the difference of any two of these sets. Let the field F' consist of another system of concentric circles of radii $1/2, 3/4, 7/8, \cdots$, the sum of these circles, and the difference of any two of these sets. If we let circles of equal radii correspond and let

the circle of radius 2 in the first system correspond to the sum of the sets in the second, the two systems will be isomorphic but the first does not contain the sum of the countable number of circles of radii less than one. Needless to add, the limit sets [18] of topologically convergent sequences of sets of the systems may not belong to either system, much less correspond even when the sets of the sequences do.

Hence we may consider the following questions. Given two isomorphic fields, under what conditions does the sum or intersection of a countable sequence of sets belong to the second field if the sum or intersection of the corresponding sets belongs to the first? Under what conditions do these sums or intersections correspond when they belong? Under what conditions do the limit sets of convergent sequences of corresponding sets belong to the fields and correspond? These questions are considered in this part; part of the first question and others are more conveniently reserved for the next one.

Essentially the problems which follow are concerned with conditions under which the isomorphism between two fields can be extended to new elements of the fields. The theorems hold in separable, metric spaces unless otherwise indicated.

DEFINITIONS. *A σ-ring is a ring to which the sum of each (countable) sequence of sets of the ring belongs. Similarly for σ-fields.*[19] *By a complete σ-field we shall mean a σ-field with unit element.*

THEOREM XII. *Let the complete σ-field F' be the isomorph of the complete σ-field F. Let $\{A_n\}$ be any sequence of sets of F and $\{A'_n\}$ the sequence of corresponding sets of F'. Then $\sum_{n=1}^{\infty} A_n$ corresponds to $\sum_{n=1}^{\infty} A'_n$; $\prod_{n=1}^{\infty} A_n$ and $\prod_{n=1}^{\infty} A'_n$ belong to F and F' respectively and correspond.*

Proof. Let $A = \sum_{n=1}^{\infty} A_n$ of F correspond to A' of F'. Also, let $\sum_{n=1}^{\infty} A'_n$ of F' correspond to B of F. Since $A \cdot A_n = A_n$, $A' \cdot A'_n = A'_n$ and therefore $A' \supset \sum_{n=1}^{\infty} A'_n$. Likewise, $B \supset \sum_{n=1}^{\infty} A_n$. However, this last implies that $\sum_{n=1}^{\infty} A'_n \supset A'$. Hence $A' = \sum_{n=1}^{\infty} A'_n$ and $B = \sum_{n=1}^{\infty} A_n$. This proves the first part of the theorem.

If A_n of F corresponds to A'_n of F', then $C(A_n)$ corresponds to $C(A'_n)$

[18] The term limit set will be used to mean the limit set of a sequence of point sets converging in the topological sense. See Kuratowski, *loc. cit.*, p. 155.

[19] F. Hausdorff, *Grundzüge der Mengenlehre*, p. 23.

by the corollary to theorem VII. Then $\sum_{n=1}^{\infty} C(A_n)$ and $\sum_{n=1}^{\infty} C(A'_n)$ belong to F and F' respectively and they correspond to each other by the first part of this theorem. Since $\prod_{n=1}^{\infty} A_n = C(\sum_{n=1}^{\infty} C(A_n))$ and similarly for $\prod_{n=1}^{\infty} A'_n$, the rest of the theorem follows.

THEOREM XIII. *Let the complete σ-field F' be the isomorph of the complete σ-field F so that closed sets of F correspond to closed sets of F' and conversely. If the limit sets of monotonically increasing sequences of sets of F and F' belong to F and F' respectively, then the limit sets of all convergent sequences of sets of F and F' belong to F and F' respectively and correspond when the sets of the sequences do.*

Proof. We first prove that the limit sets of monotonically increasing sequences of corresponding sets correspond. Let $A_1 \subset A_2 \subset A_3 \subset \cdots$ be any monotonically increasing sequence of sets of F. The limit of this sequence is $\overline{\sum_{n=1}^{\infty} A_n}$.[20] The sequence of corresponding sets $\{A'_n\}$ is such that $A'_1 \subset A'_2 \subset A'_3 \subset \cdots$ and has for its limit $\overline{\sum_{n=1}^{\infty} A'_n}$. Suppose $\overline{\sum_{n=1}^{\infty} A_n}$ corresponds to A' of F' and $\overline{\sum_{n=1}^{\infty} A'_n}$ corresponds to B of F. Since $\overline{\sum_{n=1}^{\infty} A_n} \supset \sum_{n=1}^{\infty} A'_n$, $A' \supset \sum_{n=1}^{\infty} A'_n$; and, since by hypothesis A' is closed, $A' \supset \overline{\sum_{n=1}^{\infty} A'_n}$. Likewise $B \supset \overline{\sum_{n=1}^{\infty} A_n}$. Then, by the isomorphism, $\overline{\sum_{n=1}^{\infty} A'_n} \supset A'$. Hence $A' = \overline{\sum_{n=1}^{\infty} A'_n}$ and $B = \overline{\sum_{n=1}^{\infty} A_n}$.

We now note that if A is any set of F and A' the corresponding set of F', then $\bar{A}$, the closure of A, belongs to F, and corresponds to $\bar{A}'$, which belongs to F'. For, if we form the sequence for which $A_n = A$, then $\lim_{n=\infty} A_n = \bar{A}$. Likewise for $\bar{A}'$ and the first part of the proof gives the fact that $\bar{A}$ corresponds to $\bar{A}'$.

Now let $\{A_n\}$ be a sequence of monotonically decreasing sets of F and let $\{A'_n\}$ be the sequence of corresponding sets of F'. Then $\lim_{n=\infty} A_n$ belongs to F; $\lim_{n=\infty} A'_n$ belongs to F'; and the first limit corresponds to the second. For, $\lim_{n=\infty} A_n = \prod_{n=1}^{\infty} \bar{A}_n$ [21] and $\lim_{n=\infty} A'_n = \prod_{n=1}^{\infty} \bar{A}'_n$. By the preceding paragraph, $\bar{A}_n$

[20] Kuratowski, *loc. cit.*, p. 155.

and $\bar{A}'_n$ belong to F and F' respectively and correspond. The preceding theorem gives the conclusions asserted in this paragraph.

To complete the proof we utilize the following elementary lemma which is stated without proof.

LEMMA. *If a sequence of sets has a limit, that limit is the limit of a monotonically decreasing sequence of sets formed from the sets of the first sequence by summing.*

To conclude the proof of the theorem we have but to note that if $\{A_n\}$ is any sequence of sets of F and $\{A'_n\}$ the sequence of corresponding sets of F', then the set $B_n = \sum_{k=n}^{\infty} A_k$ corresponds to $B'_n = \sum_{k=n}^{\infty} A'_k$ by theorem XII. By the lemma, $\lim_{n=\infty} A_n = \lim_{n=\infty} B_n$ and likewise for A'_n and B'_n. But $\lim_{n=\infty} B_n$ belongs to F and $\lim_{n=\infty} B'_n$ belongs to F' and the two limits correspond by the results of one of the paragraphs above.

The preceding theorem uses an hypothesis on the correspondence of closed sets which is, in a sense, a generalization of homeomorphism for two spaces. This is apparent if we specialize F and F' to the case where each consists of all the subsets of a given space. The question of for what isomorphic fields this condition on the sets implies a homeomorphism between the unit elements of the fields remains open as does the question of what additional conditions are necessary to imply this conclusion for any two isomorphic, complete σ-fields.

4. The correspondence of open and closed sets. This section will consider primarily conditions under which open and closed sets in one field correspond to open and closed sets in a homomorphic or isomorphic field.

THEOREM XIV. *Let F' be a complete field which is the homomorph of the complete field F. Suppose that limit sets of monotonically increasing (or decreasing) sequences of corresponding sets of F and F' belong to F and F' respectively and correspond. If A is a set of F open in the unit element S of F, then A' is open in S' of F'.*

Proof. Let $C_n = C(A)$.[22] Then $\{C_n\}$ is a monotonically increasing sequence of sets with $\lim_{n=\infty} C_n = \overline{C(A)}$. By the corollary to theorem VII, $C(A)$ corresponds to $C(A')$ and we have, by the hypothesis on limit sets, that $\overline{C(A)}$ corresponds to $\overline{C(A')}$ because the latter is the limit of the sequence for which $C'_n = C(A')$. Since A is open in S, $C(A)$ is closed in S. Hence

[21] *Ibid.*

[22] It is to be remembered in this theorem that complements are with respect to the unit elements of the complete fields.

$C(A) = \overline{C(A)}$. Then $C(A') = \overline{C(A')}$, and $C(A')$ is closed in S' or A' is open in S'.

THEOREM XV. *Let F' be a field which is the homomorph of the field F. Suppose the limit sets of monotonically decreasing sequences of corresponding sets of F and F' belong to F and F' respectively, and correspond. Then if $\{A_n\}$ is a sequence of open sets of F such that $\sum_{n=1}^{\infty} A_n$ belongs to F, then $\sum_{n=1}^{\infty} A'_n$ belongs to F', and $\sum_{n=1}^{\infty} A_n$ corresponds to $\sum_{n=1}^{\infty} A'_n$.*

Proof. Let $\sum_{n=1}^{\infty} A_n = A$. Suppose A corresponds to A'. Since $A \cdot A_n = A_n$, $A' \cdot A'_n = A'_n$ and $A' \supset \sum_{n=1}^{\infty} A'_n = B'$. We shall show that $A' - B' = 0$.

Let $C'_n = A' - \sum_{i=1}^{n} A'_i$; then $\{C'_n\}$ is a monotonically decreasing sequence of sets of F' whose limit exists and is $\prod_{n=1}^{\infty} \overline{(A' - \sum_{i=1}^{n} A'_i)}$. Call this limit C'. Since $A' - B' = \prod_{n=1}^{\infty} (A' - \sum_{i=1}^{n} A'_i)$, $C' \supset A' - B'$. A sequence in F corresponding to $\{C'_n\}$ is $\{C_n\} = \{A - \sum_{i=1}^{n} A_i\}$, and this sequence has as limit $C = \prod_{n=1}^{\infty} \overline{(A - \sum_{i=1}^{n} A_i)}$. In view of the hypothesis on limit sets C corresponds to C'. But the A_i's are open sets, as are the sets $\sum_{i=1}^{n} A_i$ for each n. Then $\overline{A - \sum_{i=1}^{n} A_i}$ contains no points of $\sum_{i=1}^{n} A_i$ for such points are interior points of $\sum_{i=1}^{n} A_i$ and hence cannot be limit points of the complement in A. Since $A = \sum_{n=1}^{\infty} A_n$, the value of C shows that $C \cdot A = 0$. In view of the homomorphism, $C' \cdot A' = 0'$. Since $C' \supset A' - B'$ it must be that $A' - B' = 0$, and the theorem follows.

COROLLARY. *Under the hypothesis of the theorem and the added condition that F be complete we have that $\sum_{n=1}^{\infty} A'_n$ is open in S', the unit element of F'.*

Proof. By the corollary to theorem VII, F' is complete. Moreover, since $\sum_{n=1}^{\infty} A_n$ is open in the space, it is open in S, the unit element of F. Then, by theorem XIV, $\sum_{n=1}^{\infty} A'_n$ is open in S'.

THEOREM XVI. *Let the complete field F' be the homomorph of the complete field F. Suppose the limit sets of monotonically decreasing sequences of corresponding sets of F and F' belong to F and F' respectively and correspond. If $\{A_n\}$ is a sequence of sets of F closed in S, the unit element of F, and such that $\prod_{n=1}^{\infty} A_n$ belongs to F, then $\prod_{n=1}^{\infty} A'_n$ belongs to F' and is the correspondent of $\prod_{n=1}^{\infty} A_n$. Moreover, $\prod_{n=1}^{\infty} A'_n$, is closed in S'.*

Proof. Since A_n is closed in S, $C(A_n)$ is open in S. Then, since $\prod_{n=1}^{\infty} A_n$ belongs to F, and since $C(\prod_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} C(A_n)$, $\sum_{n=1}^{\infty} C(A_n)$ belongs to F. By theorem XV, $\sum_{n=1}^{\infty} C(A'_n)$ belongs to F' and is the correspondent of $\sum_{n=1}^{\infty} C(A_n)$. But $\prod_{n=1}^{\infty} A'_n = C(\sum_{n=1}^{\infty} C(A'_n))$ and complements correspond under homomorphism.

By the corollary to theorem XV, $\sum_{n=1}^{\infty} C(A'_n)$ is open in S', and hence $\prod_{n=1}^{\infty} A'_n$ is closed in S'.

Several of the theorems in part III depended in part upon the hypothesis that closed sets in a field F correspond under isomorphism to closed sets in a field F'. The following theorem gives sufficient conditions for this to be the case.

THEOREM XVII. *Let F' be a field which is the isomorph of the field F. Let corresponding sets of F and F' be of the same dimension,*[23] *and, moreover, be homogeneously dimensional.*[24] *Let A of F be a set for which $\dim B(A)$*[25] *$< \dim A$, and let the same condition hold for the corresponding set A' of F'. If $\bar{A}$ and $\bar{A}'$ belong to F and F' respectively, then $\bar{A}$ corresponds to $\bar{A}'$.*

Proof. Let $\bar{A}$ of F correspond to B' of F' and let $\bar{A}'$ of F' correspond to C of F. Since $\bar{A}' \supset A'$, $C \supset A$. Likewise $B' \supset A'$. Let $C - \bar{A} = E_1$. Then $\bar{A}' - B' = E'_1$ where E'_1 is the correspondent of E_1. Suppose the dimension of A is k. Since $B(A)$ is a closed set, and hence both an F_σ and

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[23] In the sense of Menger. See his *Dimensionstheorie*, chap. II.

[24] A set is homogeneously dimensional if it is of the same dimension in each of its points.

[25] Menger defines the boundary of a set only for open sets. See his *Dimensionstheorie*, p. 34. We use the more general definition of Kuratowski, *loc. cit.*, p. 24. There the boundary of a set A is defined to be $\overline{A} \cdot \overline{C(A)}$.

a G_δ,[26] and since $\bar{A} = A + B(A)$, $\bar{A}$ is k-dimensional.[27] Then, by hypothesis, B' is k-dimensional. Since A is k-dimensional, again by hypothesis, A' is. Then, by reasoning similar to that just used, $\bar{A}'$ and C are k-dimensional.

Since $B' \supset A'$, E'_1, which is $\bar{A}' - B'$, is a subset of the boundary of A', and, since $\dim B(A') < \dim A'$, $\dim E'_1 < k$.[28] Hence, E_1 is of dimension $< k$. Since $C = C \cdot \bar{A} + E_1$, and since $\bar{A} \cdot E_1 = 0$, C must be at most $(k-1)$-dimensional in the points of E_1 because such points can be enclosed in neighborhoods whose boundaries do not intersect $\bar{A}$ and hence intersect C in points of E_1 only. However, since C is homogeneously dimensional, $E_1 = 0$. In view of the isomorphism, $E'_1 = 0$. Then $B' \supset \bar{A}'$ and $\bar{A} \supset C$. Now let $\bar{A} - C = E_2$. Then $B' - \bar{A}' = E'_2$ where E'_2 corresponds to E_2. In view of the symmetry due to the isomorphism we may use the process just employed to show that $E'_2 = 0$. Then we have that $B' \subset \bar{A}'$, and therefore that $B' = \bar{A}'$. Hence $\bar{A}$ corresponds to $\bar{A}'$.

The above theorem has application as the following two corollaries show.

Corollary I. *Let the hypothesis of the above theorem relating to F and F' hold for F and F' in the Cartesian n-dimensional space, R_n. Let A and A' be corresponding, open sets of F and F' respectively. If $\bar{A}$ belongs to F and $\bar{A}'$ to F', then $\bar{A}$ corresponds to $\bar{A}'$.*

Proof. We have but to apply the theorem[29] that in R_n the boundaries of A and A' are of dimension $\leqq n-1$.

Corollary II. *Let the hypothesis of the above theorem relating to F and F' hold in R_n, where F and F' are now σ-fields. Let $\{A_n\}$ and $\{A'_n\}$ be sequences of corresponding, open sets of F and F' respectively. If F and F' each contain the closures of their sets, then $\overline{\sum_{n=1}^{\infty} A_n}$ and $\overline{\sum_{n=1}^{\infty} A'_n}$ belong to F and F' respectively and correspond.*

Proof. That $\sum_{n=1}^{\infty} A_n$ and $\sum_{n=1}^{\infty} A'_n$ belong to F and F' respectively follows from the definition of σ-fields. They correspond to each other according to theorem XII. Then $\overline{\sum_{n=1}^{\infty} A_n}$ and $\overline{\sum_{n=1}^{\infty} A'_n}$ belong to F and F' respectively by hypothesis and they correspond to each other by the first corollary.

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[26] Hausdorff, *Grundzüge der Mengenlehre*, p. 306.

[27] Menger, *loc. cit.*, p. 114.

[28] *Ibid.*, p. 81.

[29] Menger, *loc. cit.*, p. 268.

POLYNOMIAL IDEALS DEFINED BY INFINITELY NEAR BASE POINTS.*

By Oscar Zariski.

Introduction. The linear systems of curves, in the plane or on an algebraic surface, which are theoretically of importance, are the *complete systems.* For complete linear systems the defining linear conditions are *base conditions,* by which the curves of the system are constrained to pass with assigned multiplicities through an assigned set of base points. The set of base points may consist in part of proper points and in part of points infinitely near and in the successive neighborhoods of the proper points ([10], p. 27). However vague this geometric terminology may sound, it is nevertheless true that the facts involved have a precise algebraic meaning, and satisfactory definitions are available in terms of analytical branches and of intersection multiplicities of such branches. But it is equally true that although a well rounded geometric theory can and has been developed along these lines ([1], pp. 327-399), the arithmetic content of the notion of infinitely near points still remains somewhat obscure. It is the main purpose of the present investigation to develop an arithmetic theory parallel to the geometric theory of infinitely near points (in the plane or on a surface without singularities). By this we mean primarily a systematic study of those polynomials ideals in $\mathfrak{k}[x, y]$, or formal power series ideals (in two indeterminates), which adequately describe linear conditions having the character of base conditions. We call these ideals *complete ideals* (II, 12) by analogy with the terminology used in the theory of linear systems. We always suppose that the underlying field $\mathfrak{k}$ is algebraically closed and of characteristic zero, but the theory could be extended with a few modifications to fields of any characteristic. At any rate, the hypothesis that the characteristic is zero is not used in the first four sections of Part I. For possible generalizations to spaces of higher dimension than 2 it would be important to consider also fields which are not algebraically closed.

The class of complete ideals enjoys several striking properties, which respond, however, to a high geometric expectation. *This class is closed under all standard operations on ideals* (except addition) : *the intersection, the product and the quotient of two complete ideals is a complete ideal* (II, 12). Moreover, *a complete ideal has a unique factorization into simple complete ideals* (I, 7), an ideal being simple if it is not the product of ideals different from the unit ideal.

* Received November 8, 1937.

We define complete ideals in terms of *valuation ideals.* By valuation ideals in the polynomial ring $\mathfrak{k}[x, y]$ we mean the contracted ideals of the ideals of any valuation ring (belonging to some valuation of the field $\mathfrak{k}(x, y)$) which contains x and y. Valuation ideals are complete ideals, and, in particular, the simple complete ideals are valuation ideals. Most of our work is a study of valuation ideals (briefly: v-ideals) whether from an axiomatic point of view (Part I) or from the point of view of formal power series (Part II). The study of the behaviour of valuation ideals under quadratic transformations (I, 4 and 5) leads to reduction theorems (Theorems 4.4 and 5.3) which form the basis of many inductive proofs. *A simple v-ideal of kind $k+1$ (I, 6) represents the arithmetic analogue of the notion of a point infinitely near and in the k-th neighborhood of a proper point.*

The treatment deals explicitly only with polynomial rings and rings of holomorphic functions. It is clear, however, that the results carry over automatically to algebraic surfaces without singularities, since any set of base conditions at a simple point P of an algebraic surface is described by a complete ideal in the ring of holomorphic functions of the uniformizing parameters at P.

We make one more remark. In many instances the proofs do not depend on the fact that we have only two indeterminates. On the other hand, in many points a generalization to any number of variables faces new difficulties. In spaces of higher dimension the base conditions may be of a more complicated type: besides base conditions at an isolated base point, it is possible to have infinitely near base curves, or infinitely near base surfaces, etc. At an isolated base point we may also have such base conditions as are given by infinitesimal base curves (the case of an assigned tangent plane at a base point is the simplest example). The algebro-geometric theory has as yet no firm grasp on these eventualities. A generalization of the present treatment to any number of variables would therefore represent not merely an arithmetization of a known chapter of classical algebraic geometry.

CONTENTS.

Part I.

PART II.

PART I.

1. Valuation ideals in rings of polynomials in two indeterminates. Let $\mathfrak{f}$ be an algebraically closed field of characteristic zero and Σ—a pure transcendental extension of $\mathfrak{f}$ of dimension (degree of transcendentality) two. We consider a valuation B of Σ, i. e. an homorphic mapping of the multiplicative group Σ (the element 0 excluded) upon an ordered abelian group Γ: $a \rightarrow v(a)$ $=$ *value of* a, $a \in \Sigma$, $a \neq 0$, $v(a) \in \Gamma$, satisfying the valuation axioms:

$$(1)\ v(a \cdot b) = v(a) + v(b); \qquad (2)\ v(a+b) \geqq \min.(v(a), v(b));$$
$$(3)\ v(a^*) \neq 0,$$

for some a^* in Σ ([2], p. 101; [7]).[1] We assume, moreover, that the elements of the underlying field $\mathfrak{f}$, other than 0, have value zero in the given valuation B.

Let $\mathfrak{B}$ be the valuation ring of B (the set of all elements of Σ whose value is $\geqq 0$). Any ideal $\mathfrak{a}$ in $\mathfrak{B}$ has the following self-evident property: $a \equiv 0(\mathfrak{a})$, $v(b) \geqq v(a)$ implies $b \equiv 0(\mathfrak{a})$. Conversely, any subset of $\mathfrak{B}$ with this property constitutes, together with 0, an ideal. Consequently, given any two ideals $\mathfrak{a}, \mathfrak{a}'$ in $\mathfrak{B}$, either $\mathfrak{a} \equiv 0(\mathfrak{a}')$ or $\mathfrak{a}' \equiv 0(\mathfrak{a})$. We say that $\mathfrak{a}$ *precedes* $\mathfrak{a}'$ (and that $\mathfrak{a}'$ *follows* $\mathfrak{a}$) if $\mathfrak{a}' \equiv 0(\mathfrak{a})$, but $\mathfrak{a}' \neq \mathfrak{a}$. The ideals in $\mathfrak{B}$ form then an ordered set. The unit ideal $\mathfrak{B}$ is the first element of this set. The immediate successor of $\mathfrak{B}$ is the ideal $\mathfrak{P}$ consisting of all elements whose value is positive.

Let $\mathfrak{O}$ be a domain of integrity in Σ, *contained in the valuation ring* $\mathfrak{B}$. An ideal $\mathfrak{A}$ in $\mathfrak{O}$ shall be called a *valuation ideal*, or briefly, a *v-ideal, belonging to* or *for the valuation B*, if $\mathfrak{A}$ is the contracted ideal of an ideal $\mathfrak{a}$ in $\mathfrak{B}$, i. e.

[1] Note the following consequence of the above axioms: *if* $v(a) < v(b)$, *then* $v(a+b) = v(a)$. Proof: We have $v(a) = v(a \cdot 1) = v(a) + v(1)$, whence $v(1) = 0$. Also $v(-1) + v(-1) = v((-1)^2) = v(1) = 0$, consequently $v(-1) = 0$, since Γ is an ordered group. Hence $v(-b) = v(b)$, and, by axiom (2), $v(a-b) \geqq \min.(v(a), v(b))$. Replacing in this relation a by $a+b$ we find $v(a) \geqq \min.(v(a+b), v(b))$, and our assertion follows.

if $\mathfrak{A} = [\mathfrak{O}, \mathfrak{a}]$. If the reference to the specific valuation B is omitted and if we speak of $\mathfrak{A}$ as a v-ideal, we shall mean then that $\mathfrak{A}$ is a valuation ideal for some valuation B of Σ such that the valuation ring of B contains $\mathfrak{O}$. We have thus defined in $\mathfrak{O}$ a special class of ideals, the class of valuation ideals. We agree to include the zero ideal in this class. Also the valuation ideals in $\mathfrak{O}$, *belonging to a given valuation* B, enjoy the property: $a \equiv 0(\mathfrak{A})$, $v(b) \geqq v(a)$ implies $b \equiv 0(\mathfrak{A})$, where $\mathfrak{A}$ is a v-ideal and a, b are elements of $\mathfrak{O}$. Conversely, any subset of $\mathfrak{O}$ with this property constitutes, together with the zero element of Σ, a valuation ideal belonging to B. The valuation ideals in $\mathfrak{O}$, belonging to a given valuation B, form an ordered set: $\mathfrak{A}$ precedes $\mathfrak{A}'$ if $\mathfrak{A}' \equiv 0(\mathfrak{A})$ and $\mathfrak{A}' \neq \mathfrak{A}$. Since a v-ideal $\mathfrak{A}$ in $\mathfrak{O}$, for the valuation B, is the contracted ideal of an ideal $\mathfrak{a}$ in $\mathfrak{B}$, $\mathfrak{A}$ is also the contracted ideal of its extended ideal $\mathfrak{B}\mathfrak{A}$ in $\mathfrak{B}$. There is thus a $(1,1)$ correspondence between the v-ideals in $\mathfrak{O}$ belonging to the valuation B and their extended ideals in the valuation ring $\mathfrak{B}$. These extended ideals form in general a proper subset of the set of all ideals of $\mathfrak{B}$.

Let, in particular, $\mathfrak{O}$ be the ring of polynomials in x, y, where we assume that x and y are generating elements of Σ ($\Sigma = k(x, y)$) and elements of the valuation ring.[2] From the fact that every ideal in $\mathfrak{O}$ possesses a finite base and from the valuation axiom (2), it follows that any ideal $\mathfrak{M}$ in $\mathfrak{O}$ contains elements of smallest possible value in B. If α is this minimum, $\alpha = \min\{v(a)\}$, $a \varepsilon \mathfrak{M}$, we shall write $\alpha = v(\mathfrak{M})$ and we shall regard α as *the evaluation of the ideal* $\mathfrak{M}$. Two ideals $\mathfrak{M}$, $\mathfrak{M}'$ will be said to be *equivalent*, in symbols $\mathfrak{M} \sim \mathfrak{M}'$, if $v(\mathfrak{M}) = v(\mathfrak{M}')$. The class $\{\mathfrak{M}\}$ of all ideals equivalent to $\mathfrak{M}$ contains one and only one v-ideal for B, namely the ideal $\mathfrak{A}$ consisting of all elements whose value is $\geqq v(\mathfrak{M})$. Clearly $\mathfrak{A}$ is a divisor of any ideal of the class. In the ordered set of v-ideals of $\mathfrak{O}$, belonging to the valuation B, $\mathfrak{A}$ precedes $\mathfrak{A}'$ if $v(\mathfrak{A}) < v(\mathfrak{A}')$.

In the sequel we shall be dealing with a fixed valuation B and it will be understood that when we speak of a v-ideal in $\mathfrak{O}$ we mean a v-ideal "belonging to the given valuation B."

The valuation ring $\mathfrak{B}$ contains the divisorless ideal $\mathfrak{P}$, consisting of all elements whose value is > 0. We shall denote by $\mathfrak{p}$ the contracted ideal of $\mathfrak{P}$ in $\mathfrak{O}$; $\mathfrak{p}$ is obviously a prime ideal in $\mathfrak{O}$. By the *dimension* r *of the valuation* B is meant the dimension of the ideal $\mathfrak{P}$, i. e. the degree of transcendentality of the field of residual classes $\mathfrak{B}/\mathfrak{P}$ over $\mathfrak{k}$.[3] Evidently, r is either 0 or 1.

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[2] Note that if x is any element in Σ, then either x or $1/x$ belongs to $\mathfrak{B}$, since $v(1/x) = -v(x)$.

[3] By hypothesis, 0 is the only element of $\mathfrak{k}$ which belongs to $\mathfrak{P}$. Hence $\mathfrak{B}/\mathfrak{P}$ contains a subfield isomorphic to $\mathfrak{k}$. We identify this subfield with $\mathfrak{k}$.

THEOREM 1. *In the ordered set of v-ideals in $\mathfrak{O}$ every v-ideal $\mathfrak{A}$ has an immediate successor $\mathfrak{A}'$. If the valuation B is of dimension zero, then $\mathfrak{A}'$ is a maximal subideal of $\mathfrak{A}$, and the ring $\mathfrak{A}/\mathfrak{A}'$ is isomorphic to the underlying field $\mathfrak{k}$.*

Proof. By the valuation axiom (3), there exists an element a^* such that $v(a^*) > 0$. Let $a^* = f(x,y)/g(x,y)$, $f \varepsilon \mathfrak{O}$, $g \varepsilon \mathfrak{O}$. Since $v(f) = v(a^*) + v(g)$ and $v(g) \geqq 0$, also $v(f) > 0$. If then $\alpha = v(\mathfrak{A})$, there exist in $\mathfrak{A}$ elements whose value is greater than α, for instance, the elements of $\mathfrak{A}f$. The totality of all polynomials whose value is greater than α constitutes, together with the element 0, a v-ideal $\mathfrak{A}'$, contained in $\mathfrak{A}$, and clearly there exist no v-ideals which follow $\mathfrak{A}$ (proper multiples of $\mathfrak{A}$) and precede $\mathfrak{A}'$ (proper divisors of $\mathfrak{A}'$).

Assume that B is of dimension 0, whence $\mathfrak{B}/\mathfrak{P}$ is an algebraic extension of the underlying field $\mathfrak{k}$. Since $\mathfrak{k}$ is algebraically closed, it follows that $\mathfrak{B}/\mathfrak{P}$ is isomorphic to $\mathfrak{k}$. Let f be an element of $\mathfrak{A}$ of smallest possible value, $v(f) = v(\mathfrak{A})$, and let ϕ be any other element in $\mathfrak{A}$. Since $v(\phi) \geqq v(f)$, we have $v(\phi/f) \geqq 0$, i. e. ϕ/f belongs to $\mathfrak{B}$. Hence $\phi/f \equiv c(\mathfrak{P})$, where c is in $\mathfrak{k}$, $v(\phi/f - c) = v[(\phi - cf)/f] > 0$, i. e. $v(\phi - cf) > v(f)$, and consequently $\phi - cf \equiv 0(\mathfrak{A}')$. This shows that $\mathfrak{A}/\mathfrak{A}' \simeq \mathfrak{k}$ and also that $\mathfrak{A}'$ is a maximal subideal of $\mathfrak{A}$, q. e. d.

If B is of dimension 1, it defines an homomorphism of the field Σ upon the field $\mathfrak{B}/\mathfrak{P}$ of algebraic functions of one variable, i. e. B is a "divisor" of Σ. If $\mathfrak{O}$ contains elements which *mod* $\mathfrak{p}$ are transcendental with respect to $\mathfrak{k}$, i. e. if $\mathfrak{p}$ is a 1-dimensional ideal in $\mathfrak{O}$, we are dealing with a *divisor of the first kind* with respect to $\mathfrak{O}$. The ideal $\mathfrak{p}$ is then a principal ideal, say $\mathfrak{p} = (f)$, where f is an irreducible polynomial, and the v-ideals in $\mathfrak{O}$ are the ideals $\mathfrak{p}^n = (f^n)$, $n = 0, 1, 2, \cdots$. If, however, $\mathfrak{p}$ is 0-dimensional, we are dealing with a *divisor of the second kind* with respect to $\mathfrak{O}$. The v-ideals in $\mathfrak{O}$ are in this case certain primary ideals belonging to $\mathfrak{p}$. We need not consider separately this case, because it reduces to the case of 0-dimensional valuations. In fact, we may consider an arbitrary valuation B_1 of the field $\Sigma_1 = \mathfrak{B}/\mathfrak{P}$ (a point of the Riemann surface of the field Σ_1). The given valuation B of Σ followed up by the valuation B_1 of Σ_1 defines an homomorphism of Σ upon the underlying field $\mathfrak{k}$ (together with symbol ∞), hence a 0-dimensional valuation B' of Σ. The v-ideals in $\mathfrak{O}$ belonging to B will be among the v-ideals belonging to B'.

From now on we shall only consider 0-dimensional valuations. If B is 0-dimensional, then the v-ideal $\mathfrak{p} = [\mathfrak{P}, \mathfrak{O}]$ is prime and 0-dimensional, since

it is the immediate successor of the unit ideal $\mathfrak{O}$ and since, by Theorem 1, $\mathfrak{O}/\mathfrak{p} \simeq \mathfrak{k}$. Replacing, if necessary, x and y by $x - c$, $y - d$, where $x \equiv c(\mathfrak{p})$, $y \equiv d(\mathfrak{p})$, we may assume that $x \equiv 0(\mathfrak{p})$, $y \equiv 0(\mathfrak{p})$, whence $\mathfrak{p} = (x, y)$.

Starting with $\mathfrak{O} = \mathfrak{q}_0$ and with its successor $\mathfrak{p} = \mathfrak{q}_1$, we form the simple sequence of v-ideals

(1) $$\mathfrak{q}_0, \mathfrak{q}_1, \mathfrak{q}_2, \cdots, \mathfrak{q}_i, \cdots$$

where $\mathfrak{q}_{i+1}$ is the immediate successor of $\mathfrak{q}_i$. *Each* $\mathfrak{q}_{i+1}$ *is a maximal subideal of its predecessor* $\mathfrak{q}_i$. We shall call a *Jordan sequence* any sequence of ideals having this last mentioned property. It is clear that all the ideals $\mathfrak{q}_i$, $i \geqq 1$, in the Jordan sequence (1), are primary ideals belonging to $\mathfrak{p}$ $(= \mathfrak{q}_1)$. In fact: (1) $\mathfrak{q}_i \equiv 0(\mathfrak{p})$; (2) $ab \equiv 0(\mathfrak{q})$, $a \not\equiv 0(\mathfrak{q})$ imply that $v(a) + v(b) > v(a)$, whence $v(b) > 0$ and $b \equiv 0(\mathfrak{p})$; (3) $v(\mathfrak{q}_1) = v(\mathfrak{p}) < v(\mathfrak{p}^2) < v(\mathfrak{p}^3) < \cdots < v(\mathfrak{p}^i)$, whence $v(\mathfrak{p}^i) \geqq v(\mathfrak{q}_i)$ and consequently $\mathfrak{p}^i \equiv 0(\mathfrak{q}_i)$.

Two cases are possible: (a) either the intersection of all the ideals $\mathfrak{q}_i$ is the zero-ideal; (b) or this intersection is a certain ideal $\mathfrak{p}_1 \neq (0)$. In the first case the Jordan sequence (1) contains *all* the v-ideals of $\mathfrak{O}$ belonging to the given valuation B. We investigate now the second case.

We first prove that $\mathfrak{p}_1$ *is a prime ideal.* In fact, let $ab \equiv 0(\mathfrak{p}_1)$, $a \not\equiv 0(\mathfrak{p}_1)$, and let $\mathfrak{q}_r$ be the last ideal in the sequence $\{\mathfrak{q}_i\}$ which contains a. Then $v(a) = v(\mathfrak{q}_r)$, whence $v(a) \leqq \rho v(\mathfrak{p})$, where ρ is the exponent of the ideal $\mathfrak{q}_r$. No power $\mathfrak{p}^n$ of $\mathfrak{p}$ can belong to all the ideals $\mathfrak{q}_i$, since $\mathfrak{p}^n/\mathfrak{p}$ is of finite rank with respect to $\mathfrak{k}$. Hence, for any integer $n > 0$ there exists an integer α_n such that $v(\mathfrak{q}_{\alpha_n}) > v(\mathfrak{p}^n)$. Since $ab \equiv 0(\mathfrak{q}_i)$ for any value of i, it follows that $v(a) + v(b) > nv(\mathfrak{p})$, n arbitrary. In view of the inequality $v(a) \leqq \rho v(\mathfrak{p})$, we deduce that also $v(b) > nv(\mathfrak{p})$, n—an arbitrary integer. In particular, if $\mathfrak{q}_m$ is any ideal in the sequence $\{\mathfrak{q}_i\}$ and if ρ_m is its exponent, we will have $v(\mathfrak{q}_m) \leqq \rho_m v(\mathfrak{p}) < v(b)$, whence $b \equiv 0(\mathfrak{q}_m)$, for any m. It follows that $b \equiv 0(\mathfrak{p}_1)$, i. e. $\mathfrak{p}_1$ is a prime ideal.

The ideal $\mathfrak{p}_1$ is one-dimensional, say (f), where f is an irreducible polynomial. The inequality $v(b) > nv(\mathfrak{p})$, n arbitrary, holds for any element b of $\mathfrak{p}_1$, and this shows that the value group of B is non-archimedean (B is a "special" valuation, of rank 2. See [2], p. 113). It is not difficult to see that *all the v-ideals in* $\mathfrak{O}$ *for B are of the form* $\mathfrak{p}_1{}^m\mathfrak{q}_n$, $m, n = 0, 1, 2 \cdots$. In fact, let F_1 and F_2 be any two polynomials in $\mathfrak{O}$ and let $F_1 = f^{m_1}G_1$, $F_2 = f^{m_2}G_2$, where G_1 and G_2 are not divisible by f. If $m_1 > m_2$, then $v(F_1) > v(F_2)$, since $v(f) > v(G_2)$. If $m_1 = m_2$, then $v(F_1) > v(F_2)$ or $v(F_1) = v(F_2)$ according as $v(G_1) > v(G_2)$ or $v(G_1) = v(G_2)$. Hence the set of all polynomials whose value is not less than the value of a given polynomial f^mG

$(G \not\equiv 0(f))$ coincides with the ideal $\mathfrak{p}_1{}^m\mathfrak{q}_n$, where $v(\mathfrak{q}_n) = v(G)$. This form of the v-ideals brings out clearly the well-known decomposition of the valuation B into two valuations of rank 1 and the nature of the value group, as consisting in this case of pairs of integers. B decomposes into two valuations, B' and $\bar{B}$. B' is the one-dimensional valuation defined by the prime 1-dimensional ideal $\mathfrak{p}_1$, and its valuation ring $\mathfrak{B}'$ is the set of rational functions $\dfrac{F(x,y)}{G(x,y)}$, $G \not\equiv 0(f)$. B' maps Σ upon the field $\bar{\Sigma} = \mathfrak{k}(\bar{x}, \bar{y})$ of algebraic functions of one variable, where $\bar{\Sigma}$ is the quotient field of the ring $\mathfrak{O}/\mathfrak{p}_1$. $\bar{B}$ is a valuation of $\bar{\Sigma}$, and the sequence $\{\mathfrak{q}_i\}$ (with x, y replaced by $\bar{x}, \bar{y}$) is the sequence of the valuation ideals in the ring $\bar{\mathfrak{O}} = \mathfrak{k}[\bar{x}, \bar{y}]$ which belong to $\bar{B}$.[4]

2. A characteristic property of Jordan sequence of valuation ideals. Given a Jordan sequence of ideals in $\mathfrak{O}$: $\mathfrak{q}_0, \mathfrak{q}_1, \mathfrak{q}_2, \cdots$, where $\mathfrak{q}_0 = \mathfrak{O}$, $\mathfrak{q}_1 = \mathfrak{p} = (x, y)$, we ask under what conditions will there exist a valuation of Σ for which the given sequence $\{\mathfrak{q}_i\}$ is the sequence of (zero-dimensional) v-ideals. In other words: *under what conditions does the given sequence $\{\mathfrak{q}_i\}$ belong to a valuation of Σ?*

THEOREM 2. 1. *A necessary and sufficient condition in order that a Jordan sequence $\{\mathfrak{q}_i\}$ of 0-dimensional ideals in $\mathfrak{O}$ belong to a valuation of the field Σ, is that the quotient $\mathfrak{q}_i : (a)$ belong to the sequence, for any i and for any elements a in $\mathfrak{O}$.*

THEOREM 2. 2. *A necessary and sufficient condition in order that a Jordan sequence $\{\mathfrak{q}_i\}$ of 0-dimensional ideals in $\mathfrak{O}$ belong to a valuation of the field Σ, is that the congruences*

$$\mathfrak{q}_i\mathfrak{q}_{j+1} : \mathfrak{q}_j \equiv 0(\mathfrak{q}_{i+1}) \tag{2}$$

hold true for any pair $\mathfrak{q}_i, \mathfrak{q}_j$ of ideals of the sequence.

The characterization given in Theorem 2. 2 has the advantage of involving only operations within the given sequence $\{\mathfrak{q}_i\}$. We prove both theorems simultaneously.

(1) *The conditions are necessary.* As to Theorem 2. 1, let $v(a) = \alpha$, $\mathfrak{q}_i : (a) = \mathfrak{q}'$, and let b, c be any two elements of $\mathfrak{O}$ such that $b \equiv 0(\mathfrak{q}')$, $v(c) \geqq v(b)$. Since $ba \equiv 0(\mathfrak{q}_i)$, we have $v(c) + v(a) \geqq v(b) + v(a) \geqq v(\mathfrak{q}_i)$, whence $ca \equiv 0(\mathfrak{q}_i)$, and consequently $c \equiv 0(\mathfrak{q}')$. Hence $\mathfrak{q}'$ enjoys the property:

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[4] It is possible to have $\mathfrak{p}_1 = (0)$ also in the case of a valuation of rank 2. This happens when the component B' (divisor of Σ) of the valuation B, of rank 2, is a divisor of the second kind with respect to $\mathfrak{O}$. (In geometric terms: the divisor B' is an exceptional curve which has been transformed into a point of the plane (x, y)).

$b \equiv 0(\mathfrak{q}')$, $v(c) \geqq v(b)$, implies $c \equiv 0(\mathfrak{q}')$, and is therefore a v-ideal belonging to the given valuation B. Since $\mathfrak{q}_i \equiv 0(\mathfrak{q}')$, necessarily $\mathfrak{q}' = \mathfrak{q}_j$, for some $j \leqq i$.

The necessity of the condition of Theorem 2.2 is proved in a similar manner. Let b be an element of $\mathfrak{q}_i\mathfrak{q}_{j+1} : \mathfrak{q}_j$, whence $b\mathfrak{q}_j \equiv 0(\mathfrak{q}_i\mathfrak{q}_{j+1})$. Since $v(\mathfrak{q}_i\mathfrak{q}_{j+1}) = v(\mathfrak{q}_i) + v(\mathfrak{q}_{j+1})$, we have $v(b) + v(\mathfrak{q}_j) \geqq v(\mathfrak{q}_i) + v(\mathfrak{q}_{j+1})$, and since $v(\mathfrak{q}_j) < v(\mathfrak{q}_{j+1})$, it follows, $v(b) > v(\mathfrak{q}_i)$, i. e. $b \equiv 0(\mathfrak{q}_{i+1})$, q. e. d.

(2) *The conditions are sufficient.* We introduce the following notations: if ξ, η are elements of $\mathfrak{O}$, we write $\xi \leqq \eta$ (or $\eta \geqq \xi$), if the congruence $\xi \equiv 0(\mathfrak{q}_i)$ always implies $\eta \equiv 0(\mathfrak{q}_i)$; we write $\xi < \eta$ (or $\eta > \xi$) if there exists in the sequence $\{\mathfrak{q}_i\}$ an ideal $\mathfrak{q}_m$ such that $\eta \equiv 0(\mathfrak{q}_m)$, $\xi \not\equiv 0(\mathfrak{q}_m)$. We now prove the following lemma, assuming that the condition of Theorem 2.1 *or* that of Theorem 2.2 is satisfied.

LEMMA. *If ξ, η, ζ are elements of $\mathfrak{O}$ and if $\xi\eta \equiv 0(\mathfrak{q}_i)$, then $\eta \leqq \zeta$ implies $\xi\zeta \equiv 0(\mathfrak{q}_i)$ and $\eta < \zeta$ implies $\xi\zeta \equiv 0(\mathfrak{q}_{i+1})$.*

In other words: *$\eta \leqq \zeta$ implies $\xi\eta \leqq \xi\zeta$; $\eta < \zeta$ implies that either $\xi\eta < \xi\zeta$ or that $\xi\eta$ and $\xi\zeta$ belong to all the ideals $\mathfrak{q}_i$ of the sequence.*

Assume the condition of Theorem 2.1, and let $\mathfrak{q}_i : (\xi) = \mathfrak{q}_h$. Since $\eta \equiv 0(\mathfrak{q}_h)$, it follows that if $\eta \leqq \zeta$, then also $\zeta \equiv 0(\mathfrak{q}_h)$, whence $\xi\zeta \equiv 0(\mathfrak{q}_i)$, and this proves the first part of the lemma. Let $\mathfrak{q}_{i+1} : (\xi) = \mathfrak{q}_s$, $s \geqq h$, and let ϕ and ψ be any pair of elements of $\mathfrak{q}_h$. We have $\phi\xi \equiv 0(\mathfrak{q}_i)$, $\psi\xi \equiv 0(\mathfrak{q}_i)$, whence there exist elements c, d in the underlying field $\mathfrak{k}$ such that $c\phi\xi + d\psi\xi \equiv 0(\mathfrak{q}_{i+1})$, since $\mathfrak{q}_i/\mathfrak{q}_{i+1}$ is of rank 1 with respect to $\mathfrak{k}$. Hence $c\phi + d\psi \equiv 0(\mathfrak{q}_s)$, for any two elements ϕ, ψ in $\mathfrak{q}_h$ and for appropriate elements c, d in $\mathfrak{k}$, i. e. $\mathfrak{q}_h/\mathfrak{q}_s$ is at most of rank 1 with respect to $\mathfrak{k}$, and s is either h or $h+1$. Since $\eta \equiv 0(\mathfrak{q}_h)$, it follows that if $\eta < \zeta$, then $\zeta \equiv 0(\mathfrak{q}_{h+1}) = 0(\mathfrak{q}_s)$, whence $\xi\zeta = 0(\mathfrak{q}_{i+1})$, q. e. d.

Assume the condition of Theorem 2.2. If η belongs to all the ideals $\mathfrak{q}_i$ of the sequence, the same will be true for ζ, and both parts of the lemma are trivial. Assume that there exists a *last* ideal $\mathfrak{q}_h$ which contains η: $\eta \equiv 0(\mathfrak{q}_h)$, $\eta \not\equiv 0(\mathfrak{q}_{h+1})$. If $\eta \leqq \zeta$, then also $\zeta \equiv 0(\mathfrak{q}_h)$, whence $\xi\eta\zeta \equiv 0(\mathfrak{q}_i\mathfrak{q}_h)$. Assume, if possible, $\xi\zeta \not\equiv 0(\mathfrak{q}_i)$. There will then exist an ideal $\mathfrak{q}_j$ in our sequence, $j < i$, such that $\xi\zeta \equiv 0(\mathfrak{q}_j)$, $\xi\zeta \not\equiv 0(\mathfrak{q}_{j+1})$. Since $j + 1 \leqq i$, $\mathfrak{q}_i \equiv 0(\mathfrak{q}_{j+1})$, the congruence $\xi\eta\zeta \equiv 0(\mathfrak{q}_i\mathfrak{q}_h)$ implies $\xi\eta\zeta \equiv 0(\mathfrak{q}_{j+1}\mathfrak{q}_h)$. Now $\mathfrak{q}_{j+1}$ is a maximal subideal of $\mathfrak{q}_j$ and $\xi\zeta$ is in $\mathfrak{q}_j$ but not in $\mathfrak{q}_{j+1}$; hence $\mathfrak{q}_j = (\xi\zeta, \mathfrak{q}_{j+1})$, and consequently, $\eta\mathfrak{q}_j = (\eta\xi\zeta, \eta\mathfrak{q}_{j+1}) \equiv 0(\mathfrak{q}_{j+1}\mathfrak{q}_h)$, since $\eta \equiv 0(\mathfrak{q}_h)$. It follows that $\eta \equiv 0(\mathfrak{q}_{j+1}\mathfrak{q}_h : \mathfrak{q}_j) \equiv 0(\mathfrak{q}_{h+1})$, in contradiction with our hypothesis $\eta \not\equiv 0(\mathfrak{q}_{h+1})$. This proves the first part of the lemma.

Let now $\eta < \zeta$, and hence $\zeta \equiv 0(\mathfrak{q}_{h+1})$. We have then $\xi\eta\zeta \equiv 0(\mathfrak{q}_i\mathfrak{q}_{h+1})$

and also by the first part of the lemma, just proved, $\xi\zeta \equiv 0(\mathfrak{q}_i)$. Since $\mathfrak{q}_{h+1}$ is a maximal subideal of $\mathfrak{q}_h$ and since η is in $\mathfrak{q}_h$ but not in $\mathfrak{q}_{h+1}$, we have $\mathfrak{q}_h = (\eta, \mathfrak{q}_{h+1})$, and consequently $\xi\zeta\mathfrak{q}_h = (\xi\zeta\eta, \xi\zeta\mathfrak{q}_{h+1}) \equiv 0(\mathfrak{q}_i\mathfrak{q}_{h+1})$. Hence $\xi\zeta \equiv 0(\mathfrak{q}_i\mathfrak{q}_{h+1} : \mathfrak{q}_h) \equiv 0(\mathfrak{q}_{i+1})$, q. e. d.

The rest of the proof is based solely on the above Lemma. Let $\mathfrak{p}_1$ be the intersection of the ideals $\mathfrak{q}_i$ of our sequence. *We prove that $\mathfrak{p}_1$ is a prime ideal.* We first observe, that x and y cannot both belong to $\mathfrak{q}_2$, since $\mathfrak{q}_1 = \mathfrak{p} = (x,y)$. Let, for instance, $x \equiv 0(\mathfrak{q}_1)$, $x \not\equiv 0(\mathfrak{q}_2)$, whence $y \geqq x$. We deduce from our lemma, that if a given power of x, say x^m, belongs to an ideal $\mathfrak{q}_i$ of the sequence, then also all the power products x^ky^l, $k + l \geqq m$, belong to $\mathfrak{q}_i$, i. e. $\mathfrak{p}^m \equiv 0(\mathfrak{q}_i)$. Since $\mathfrak{p}^m/\mathfrak{p}$ is of finite rank with respect to $\mathfrak{k}$, it follows that *no power of x can belong to all the ideals $\mathfrak{q}_i$, i. e.* to $\mathfrak{p}_1$.

We next observe that *all the ideals $\mathfrak{q}_i$, $i > 1$, are primary ideals belonging to $\mathfrak{p}$ $(= \mathfrak{q}_1)$.* In fact: (1) $\mathfrak{q}_i \equiv 0(\mathfrak{p})$. (2) Let $ab \equiv 0(\mathfrak{q}_i)$, $a \not\equiv 0(\mathfrak{q}_i)$, whence $a < ab$. It is not possible to have $b \leqq 1$, because this would imply, by our lemma, $ab \leqq a$, in contradiction with $ab > a$. Hence $1 < b$ and since 1 is in $\mathfrak{q}_0$, b must be in $\mathfrak{q}_1$, i. e. in $\mathfrak{p}$. (3) We have $1 < x$, hence, by our Lemma, $x < x^2$ (since no power of x belongs to $\mathfrak{p}_1$), whence, again by the Lemma, $x^2 < x^3$, and generally, $x < x^2 < x^3 < \cdots < x^i$. Consequently $x^2 \equiv 0(\mathfrak{q}_2)$, $x^3 \equiv 0(\mathfrak{q}_3), \cdots, x^i \equiv 0(\mathfrak{q}_i)$. The congruence $x^i \equiv 0(\mathfrak{q}_i)$ implies, as we have just shown, the congruence $\mathfrak{p}^i \equiv 0(\mathfrak{q}_i)$.

To prove that $\mathfrak{p}_1$ is prime, let $\xi\eta \equiv 0(\mathfrak{p}_1)$ and assume, if possible, that $\xi \not\equiv 0(\mathfrak{p}_1)$, $\eta \not\equiv 0(\mathfrak{p}_1)$. Let $\mathfrak{q}_r$ be the last ideal of the sequence $\{\mathfrak{q}_i\}$ which contains ξ and let similarly $\mathfrak{q}_s$ be the last ideal containing η. If m and n are the exponents of the primary ideals $\mathfrak{q}_r$ and $\mathfrak{q}_s$ respectively, we have $x^m \equiv 0(\mathfrak{q}_r)$, $x^n \equiv 0(\mathfrak{q}_s)$, whence $x^m \geqq \xi$, $x^n \geqq \eta$, and consequently, by our lemma, $x^{m+n} \geqq \xi\eta$, i. e. $x^{m+n} \equiv 0(\mathfrak{p}_1)$, and this is impossible. Hence $\mathfrak{p}_1$ is prime, necessarily either the zero ideal or one-dimensional, since $\mathfrak{p}_1 \equiv 0(\mathfrak{p})$.

We now consider the well ordered descending set S of the ideals $\mathfrak{q}_{mn} = \mathfrak{p}_1{}^m\mathfrak{q}_n$, where clearly $\mathfrak{q}_{mn} \equiv 0(\mathfrak{q}_{m_1n_1})$ if $m > m_1$ or if $m = m_1$ and $n > n_1$. Here $\mathfrak{q}_{on} = \mathfrak{q}_n$ and if $\mathfrak{p}_1 = (0)$ it is understood that S coincides with the sequence $\{\mathfrak{q}_n\}$. In either case, the intersection of all the ideals $\mathfrak{q}_{mn}$ of the set is the zero ideal. Hence, given any element ξ in $\mathfrak{O}$, there will exist a *first* ideal in the set S, say $\mathfrak{q}_{ij}$, which does not contain ξ. Let η be another element in $\mathfrak{O}$ and let $\mathfrak{q}_{i'j'}$ be the first ideal in the set which does not contain η. We complete and modify our notations $\xi \leqq \eta$, $\xi < \eta$ introduced above, as follows: we write $\xi \not> \eta$ if $\mathfrak{q}_{i'j'} \equiv 0(\mathfrak{q}_{ij})$, and $\xi < \eta$ if $\mathfrak{q}_{i'j'} \equiv 0(\mathfrak{q}_{ij})$ and $\mathfrak{q}_{i'j'} \neq \mathfrak{q}_{ij}$. It is obvious that if $\xi \not> \eta$ and $\eta \not> \zeta$, then $\xi \not> \zeta$, and if $\xi < \eta$ and $\eta \not> \zeta$, or if $\xi \not> \eta$ and $\eta < \zeta$, then $\xi < \zeta$. From our Lemma and from the fact that $\mathfrak{p}_1$ is

a principal ideal, it follows in a straight-forward manner that the relations $\xi \not> \eta$, $\xi < \eta$ imply, for any element ζ in $\mathfrak{O}$, the relations $\xi\zeta \not> \eta\zeta$, $\xi\zeta < \eta\zeta$ respectively. It is also evident that if $\xi \not> \eta$, $\xi \not> \eta'$, then $\xi \not> \eta \pm \eta'$, and if $\xi < \eta$ and $\xi < \eta'$ then $\xi < \eta \pm \eta'$.

Let $\mathfrak{B}$ be the set of all elements in the field Σ which can be put in the form $\eta/\xi, \eta, \xi \varepsilon \mathfrak{O}, \xi \not> \eta$. *We prove that* $\mathfrak{B}$ *is a valuation ring.*

First, $\mathfrak{B}$ *is a ring.* In fact, let $\eta/\xi \varepsilon \mathfrak{B}$, $\eta_1/\xi_1 \varepsilon \mathfrak{B}$. Since $\xi \not> \eta$ and $\xi_1 \not> \eta_1$, we have $\xi\xi_1 \not> \eta\xi_1$ and $\xi\xi_1 \not> \xi\eta_1$, whence $\xi\xi_1 \not> \eta\xi_1 \pm \xi\eta_1$, i. e.

$$\frac{\eta}{\xi} \pm \frac{\eta_1}{\xi_1} = \frac{\eta\xi_1 \pm \xi\eta_1}{\xi\xi_1} \varepsilon \mathfrak{B}.$$

Also, since $\xi\xi_1 \not> \xi\eta_1 \not> \eta\eta_1$, the product $\eta\eta_1/\xi\xi_1$ belongs to $\mathfrak{B}$.

To prove that $\mathfrak{B}$ is a valuation ring, it is sufficient to show that given any two elements a, b in $\mathfrak{B}$, then either ab^{-1} or $a^{-1}b$ is in $\mathfrak{B}$ ([2], p. 102). In other words, we have to show that given any two elements ξ, η in $\mathfrak{O}$, either ξ/η or η/ξ must belong to $\mathfrak{B}$. But this is obvious, since one of the two relations $\xi \not> \eta$, $\eta \not> \xi$ must hold true.

Finally, the valuation abstractly defined by the valuation ring $\mathfrak{B}$ is not the trivial one, in which every element has value zero. In other words, the ring $\mathfrak{B}$ does not contain all the elements of the field Σ. In fact, take two elements ξ, η in $\mathfrak{O}$ such that definitely, $\xi < \eta$. We assert that ξ/η does not belong to $\mathfrak{B}$. Assuming the contrary, we must be able to put ξ/η in the form η_1/ξ_1, where $\xi_1 \not> \eta_1$ and $\xi\xi_1 = \eta\eta_1$. By our lemma, $\xi_1 \not> \eta_1$ implies $\eta\xi_1 \not> \eta\eta_1$, while $\xi < \eta$ implies $\xi\xi_1 < \eta\xi_1$, i. e. $\eta\eta_1 < \eta\xi_1$, giving two contradictory relations.

It remains to show that the ideals $\mathfrak{q}_{mn}$ of our well ordered set S are the v-ideals belonging to the valuation B defined by the valuation ring $\mathfrak{B}$. Consider any ideal $\mathfrak{q}_{mn}$ and let ξ, η be elements in $\mathfrak{O}$ such that $\xi \equiv 0(\mathfrak{q}_{mn})$, $v(\eta) \geqq v(\xi)$. Since $v(\eta/\xi) \geqq 0$, we must have $\eta/\xi = \eta_1/\xi_1$, where $\xi_1 \not> \eta_1$. Hence $\xi\xi_1 \not> \xi\eta_1$, i. e. $\xi\xi_1 \not> \eta\xi_1$, and this implies $\xi \not> \eta$. Hence $\eta \equiv 0(\mathfrak{q}_{mn})$, and thus $\mathfrak{q}_{mn}$ enjoys the property: $\xi \equiv 0(\mathfrak{q}_{mn}), v(\eta) \geqq v(\xi)$ implies $\eta \equiv 0(\mathfrak{q}_{mn})$. Hence $\mathfrak{q}_{mn}$ is a v-ideal belonging to the valuation B. That the set $S = \{\mathfrak{q}_{mn}\}$ contains all the v-ideals for B, is implied by the fact that each $\mathfrak{q}_{mn}$ $(n \neq 0)$ is a maximal subideal of its immediate predecessor $\mathfrak{q}_{m,n-1}$, and that $\mathfrak{q}_{m0}$ is the intersection of the ideals which precede it in S.

3. Further properties of v-ideals. We consider the sequence $\{\mathfrak{q}_i\}$ of 0-dimensional v-ideals in $\mathfrak{O}$ belonging to a fixed valuation B of Σ, where $\mathfrak{q}_1 = \mathfrak{p} = (x, y)$ and $\mathfrak{q}_0 = \mathfrak{O}$. We may assume $x \not\equiv 0(\mathfrak{q}_2)$. Since $\mathfrak{q}_1/\mathfrak{q}_2$ is of rank 1 with respect to $\mathfrak{k}$, there exists elements c, d in $\mathfrak{k}$ such that $cx + dy \equiv 0(\mathfrak{q}_2)$, $d \neq 0$. We then replace y by $cx + dy$ and thus we may assume $y \equiv 0(\mathfrak{q}_2)$, whence $\mathfrak{q}_2 = (y, x^2)$.

Let $\mathfrak{p}^h$ be the highest power of $\mathfrak{p}$ which divides a given v-ideal $\mathfrak{q}_i$; $\mathfrak{q}_i \equiv 0(\mathfrak{p}^h)$, $\mathfrak{q}_i \not\equiv 0(\mathfrak{p}^{h+1})$. Every polynomial f in $\mathfrak{q}_i$ is then of the form $f = f_h + f_{h+1}, \cdots$, where f_i is homogeneous of degree i in x and y. We shall call f_h the *subform* of f (for particular polynomials in $\mathfrak{q}_i$, f_h may be identically zero). As f varies in $\mathfrak{q}_i$, its subform f_h generates a linear system of forms, of a certain dimension $r \geqq 0$ (a $\mathfrak{k}$-module of rank $r+1$). Denote this system by $\Omega(\mathfrak{q}_i)$. We define in a similar manner the symbol $\Omega(\mathfrak{A})$ for any ideal $\mathfrak{A}$ in $\mathfrak{O}$.

THEOREM 3. *Let $\mathfrak{q}_i \equiv 0(\mathfrak{p}^h)$, $\mathfrak{q}_i \not\equiv 0(\mathfrak{p}^{h+1})$ and let $\mathfrak{A} = [\mathfrak{q}_i, \mathfrak{p}^k]$, $k \geqq h$. If $\Omega(\mathfrak{A})$ is of dimension r, then $\Omega(\mathfrak{A})$ coincides with the system of forms (of degree k) which are divisible by y^{k-r}, and, moreover, there exists a v-ideal $\mathfrak{q}_j$ in the sequence $\{\mathfrak{q}_n\}$ such that $\mathfrak{A} = \mathfrak{p}^r\mathfrak{q}_j$.*

Proof. Since $\mathfrak{q}_i$ contains polynomials ϕ whose subforms are exactly of degree h, it also contains polynomials (such as $x^{k-h}\phi$) whose subforms are of degree k. Hence $\mathfrak{A} \not\equiv 0(\mathfrak{p}^{k+1})$, and $\Omega(\mathfrak{A})$ consists of forms of degree k. Let $f = f_k + f_{k+1} + \cdots$ be a polynomial belonging to $\mathfrak{A}$ such that $f_k \not= 0$, and let $f_k = y^\rho \psi_{k-\rho}$, $\psi_{k-\rho} \not\equiv 0(y)$. Since y^ρ and $\psi_{k-\rho}$ are relatively prime, every form g, of degree $m \geqq k-1$, can be expressed as a linear combination $A\psi_{k-\rho} + By^\rho$, where A and B are forms of degree $m-k$ and $m-\rho$ respectively. It follows that given any integer $n \geqq 0$, it is possible to find two polynomials $P^{(n)}(x, y)$, $Q^{(n)}(x, y)$ of the form

$$
\begin{aligned}
P^{(n)}(x, y) &= y^\rho + A_{\rho+1}(x, y) + \cdots + A_{\rho+n}(x, y), \\
Q^{(n)}(x, y) &= \psi_{k-\rho} + B_{k-\rho+1}(x, y) + \cdots + B_{k-\rho+n}(x, y),
\end{aligned} \tag{3}
$$

where A_i, B_i are forms of degree i, in such a manner as to have

$$
f \equiv P^{(n)} Q^{(n)} (\mathfrak{p}^{k+n+1}). \tag{4}
$$

In fact, we have for the unknown form A_i, B_i the equations

$$
A_{\rho+i}\psi_{k-\rho} + B_{k-\rho+i}y^\rho = f_{k+i} - \sum_{j=1}^{i-1} A_{\rho+j}B_{k-\rho+i-j}, \qquad (i = 1, 2, \cdots, n),
$$

and these equations can be solved successively for $A_{\rho+1}$, $B_{k-\rho+1}$; $A_{\rho+2}$, $B_{k-\rho+2}$; etc. We take n sufficiently high, so as to have $\mathfrak{p}^{k+n+1} \equiv 0(\mathfrak{A})$.[5] For such a value of n we will have

$$
P^{(n)} Q^{(n)} \equiv 0(\mathfrak{A}). \tag{5}
$$

[5] Since $\mathfrak{A} = [\mathfrak{q}_i, \mathfrak{p}^k]$ and $\mathfrak{q}_i$ is a primary ideal belonging to $\mathfrak{p}$, also $\mathfrak{A}$ is a primary ideal belonging to $\mathfrak{p}$.

This implies $P^{(n)}Q^{(n)} \equiv 0(\mathfrak{q}_i)$. Now the subform $\psi_{k-\rho}$ of $Q^{(n)}$ is not divisible by y and, by our choice of the variables x, y, we have $v(y) > v(x)$. Hence $v(Q^{(n)}) = v(x^{k-\rho})$, and consequently, if $\phi_{k-\rho}$ is *any* form of degree $k-\rho$ in x, y, we have $v(\phi_{k-\rho}) \geqq v(Q^n)$. Consequently, $v(P^{(n)}\phi_{k-\rho}) \geqq v(P^{(n)}Q^{(n)})$, and, since $P^{(n)}Q^{(n)} \equiv 0(\mathfrak{q}_i)$, also $P^{(n)}\phi_{k-\rho} \equiv 0(\mathfrak{q}_i)$. This congruence holds true for any form $\phi_{k-\rho}$ of degree $k-\rho$, whence $P^{(n)}\mathfrak{p}^{k-\rho} \equiv 0(\mathfrak{q}_i)$. Since the subform $y^\rho\psi_{k-\rho}$ of $P^n\phi_{k-\rho}$ is of degree k, we have $P^{(n)}\phi_{k-\rho} \equiv 0(\mathfrak{p}^k)$, consequently, since $\mathfrak{A} = [\mathfrak{q}_i, \mathfrak{p}^k]$,

$$P^{(n)}\phi_{k-\rho} \equiv 0(\mathfrak{A}) \tag{6}$$

and

$$P^{(n)}\mathfrak{p}^{k-\rho} \equiv 0(\mathfrak{A}). \tag{6'}$$

We have then, in view of (6), that if $\mathfrak{A}$ contains a polynomial f whose subform f_k is divisible by y^ρ but not by $y^{\rho+1}$, $\mathfrak{A}$ also contains polynomials whose subforms are of degree k and are arbitrarily assigned forms divisible by y^ρ. This shows that $\Omega(\mathfrak{A})$ consists of all the forms which are divisible by a certain power of y, say y^ρ, where necessarily $\rho = k - r$, if r is the dimension of $\Omega(\mathfrak{A})$. This proves the first part of the theorem.

Let $\mathfrak{A}' = \mathfrak{A} : \mathfrak{p}^r$ and let $\mathfrak{A}' \sim \mathfrak{q}_j$, i.e. let $\mathfrak{q}_j$ be the v-ideal such that $v(\mathfrak{A}') = v(\mathfrak{q}_j)$. There exists such an ideal $\mathfrak{q}_j$ *in the sequence* $\{\mathfrak{q}_n\}$, since $\mathfrak{A}$, a primary 0-dimensional ideal, cannot belong to all the ideals $\mathfrak{q}_n$ (whose intersection $\mathfrak{p}_1$ is at least one-dimensional) and since $\mathfrak{A} \equiv 0(\mathfrak{A}')$. We have $\mathfrak{A}'\mathfrak{p}^r \equiv 0(\mathfrak{A}) \equiv 0(\mathfrak{q}_i)$, whence $\mathfrak{q}_j\mathfrak{p}^r \equiv 0(\mathfrak{q}_i)$, since $v(\mathfrak{q}_j\mathfrak{p}^r) = v(\mathfrak{A}'\mathfrak{p}^r) \geqq v(\mathfrak{q}_i)$. *We assert that* $\mathfrak{q}_j\mathfrak{p}^r$ *is also contained in the ideal* $\mathfrak{p}^k$, i.e. $\mathfrak{q}_j$ is contained in $\mathfrak{p}^{k-r}$. In fact, assume the contrary. There will then exist in $\mathfrak{q}_j$ a polynomial $F = F_\sigma + F_{\sigma+1} + \cdots$, whose subform F_σ is of degree $\sigma < k - r$, i.e. $\sigma < \rho$. The polynomial $x^{k-\sigma}F$ belongs to $\mathfrak{q}_i$, since $\mathfrak{q}_j\mathfrak{p}^r \equiv 0(\mathfrak{q}_i)$ and $k - \sigma > r$. It also belongs to $\mathfrak{p}^k$. Hence $x^{k-\sigma}F \equiv 0(\mathfrak{A})$, and this is impossible, since the subform $x^{k-\sigma}F_\sigma$ of $x^{k-\sigma}F$ is of degree k and is at most divisible by y^σ, $\sigma < \rho$. As a result, we have $\mathfrak{q}_j\mathfrak{p}^r \equiv 0(\mathfrak{q}_i)$ *and* $\mathfrak{q}_j\mathfrak{p}^r \equiv 0(\mathfrak{p}^k)$, whence

$$\mathfrak{q}_j\mathfrak{p}^r \equiv 0(\mathfrak{A}). \tag{7}$$

On the other hand, let f be any polynomial belonging to $\mathfrak{A}$, and let us first assume that its subform f_k is not divisible by $y^{\rho+1}$, $f_k = y^\rho\psi_{k-\rho}$, $\psi_{k-\rho} \not\equiv 0(y)$. As above, we determine the polynomials $P^{(n)}$ and $Q^{(n)}$, given by (3), so as to satisfy (4), and we again choose n sufficiently high so that $\mathfrak{p}^{k+n+1} \equiv 0(\mathfrak{A})$. Then the congruence (5) holds true and consequently also (6′), whence $P^{(n)} \equiv 0(\mathfrak{A}') \equiv 0(\mathfrak{q}_j)$, since $k - \rho = r$. Moreover, if n is sufficiently high, we will also have $\mathfrak{p}^{n+k+1} \equiv 0(\mathfrak{q}_j\mathfrak{p}^r)$. For such a value of n we deduce immediately from (4)

that $f \equiv 0(\mathfrak{q}_j\mathfrak{p}^r)$, since we have just seen that $P^{(n)}$ belongs to $\mathfrak{q}_j$ and since $Q^{(n)} \equiv 0(\mathfrak{p}^r)$. Thus, we have shown that every polynomial f in $\mathfrak{A}$ belongs to $\mathfrak{q}_j\mathfrak{p}^r$, provided the subform of f is not divisible by $y^{\rho+1}$. If the subform f_k of f is divisible by $y^{\rho+1}$, we consider in $\mathfrak{A}$ a polynomial $\bar{f} = f_k + \cdots$, such that $\bar{f}_k \not\equiv 0(y^{\rho+1})$. By the preceding result, we have $\bar{f} \equiv 0(\mathfrak{q}_j\mathfrak{p}^r)$ and also $\bar{f} + f \equiv 0(\mathfrak{q}_j\mathfrak{p}^r)$, whence again $f \equiv 0(\mathfrak{q}_j\mathfrak{p}^r)$. It is therefore proved that

$$\mathfrak{A} \equiv 0(\mathfrak{q}_j\mathfrak{p}^r),$$

and comparing with (7), we deduce

$$\mathfrak{A} = \mathfrak{q}_j\mathfrak{p}^r,$$

and this proves our theorem.

The following consequences can be drawn from Theorem 3:

COROLLARY 3.1. *$\mathfrak{A}$ cannot admit a factor $\mathfrak{p}^\sigma$ with $\sigma > r$ $(r = k - \rho)$, i.e. if $\mathfrak{A} = \mathfrak{p}^\sigma\mathfrak{A}_1$ is a product representation of $\mathfrak{A}$, then $\sigma \leqq r$.* In fact, the subforms of $\mathfrak{A}$ form then a system $\Omega(\mathfrak{A})$ of dimension $\geqq \sigma$.

COROLLARY 3.2 (special case $k = h$). *If $\mathfrak{q}_i \equiv 0(\mathfrak{p}^h)$, $\mathfrak{q}_i \not\equiv 0(\mathfrak{p}^{h+1})$ and if $\Omega(\mathfrak{q}_i)$ is of dimension r, then $\mathfrak{q}_i = \mathfrak{p}^r\mathfrak{q}_j$, where $\mathfrak{q}_j$ is an ideal in the sequence $\{\mathfrak{q}_n\}$, and $\mathfrak{q}_i$ does not admit as a factor a higher power of $\mathfrak{p}$ than $\mathfrak{p}^r$. In particular, if $r = 0$, then $\mathfrak{q}_i$ does not admit factors $\mathfrak{p}$ and every element of $\Omega(\mathfrak{q}_i)$ coincides, to within a constant factor in $\mathfrak{k}$, with y^h.*

COROLLARY 3.3. *If $\mathfrak{q}_i \equiv 0(\mathfrak{p}^h)$, $\mathfrak{q}_i \not\equiv 0(\mathfrak{p}^{h+1})$ and if $\mathfrak{p}^k\mathfrak{q}_i \sim \mathfrak{q}_m$ $(k \geqq 0)$, then*

$$[\mathfrak{q}_m, \mathfrak{p}^{h+k}] = \mathfrak{p}^k\mathfrak{q}_i.$$

In fact, let $\mathfrak{A} = [\mathfrak{q}_m, \mathfrak{p}^{h+k}]$. We have $\mathfrak{q}_m \not\equiv 0(\mathfrak{p}^{h+k+1})$, hence, by Theorem 3, $\mathfrak{A} = \mathfrak{p}^r\mathfrak{q}_j$, where $\mathfrak{q}_j$ is some v-ideal of our sequence $\{\mathfrak{q}_n\}$ and r is the dimension of $\Omega(\mathfrak{A})$. Since $\mathfrak{p}^k\mathfrak{q}_i \sim \mathfrak{q}_m$, we have $\mathfrak{p}^k\mathfrak{q}_i \equiv 0(\mathfrak{q}_m)$ and also $\mathfrak{p}^k\mathfrak{q}_i \equiv 0(\mathfrak{p}^{h+k})$, since $\mathfrak{q}_i \equiv 0(\mathfrak{p}^h)$. Consequently $\mathfrak{p}^k\mathfrak{q}_i \equiv 0(\mathfrak{A}) \equiv 0(\mathfrak{p}^r\mathfrak{q}_j)$. Since the dimension of $\Omega(\mathfrak{p}^k\mathfrak{q}_i)$ is at least k and since $\Omega(\mathfrak{p}^k\mathfrak{q}_i)$ is *a* subset of $\Omega(\mathfrak{A})$ (in view of the assumption $\mathfrak{q}_i \not\equiv 0(\mathfrak{p}^{h+1})$), it follows $k \leqq r$. Now, $\mathfrak{q}_m \sim \mathfrak{p}^k\mathfrak{q}_i$,

$$v(\mathfrak{p}^k\mathfrak{q}_i) = v(\mathfrak{q}_m) \leqq v(\mathfrak{A}) = v(\mathfrak{p}^r\mathfrak{q}_j), \quad \text{i.e.} \quad v(\mathfrak{q}_i) \leqq v(\mathfrak{p}^{r-k}\mathfrak{q}_j),$$

whence $\mathfrak{p}^{r-k}\mathfrak{q}_j \equiv 0(\mathfrak{q}_i)$ and $\mathfrak{p}^r\mathfrak{q}_j \equiv 0(\mathfrak{p}^k\mathfrak{q}_i)$. Since we also have $\mathfrak{p}^k\mathfrak{q}_i \equiv 0(\mathfrak{p}^r\mathfrak{q}_j)$, it follows that $\mathfrak{A} = \mathfrak{p}^r\mathfrak{q}_j = \mathfrak{p}^k\mathfrak{q}_i$, *q. e. d.*

4. v-ideals and quadratic transformations. We consider the quadratic transformation T:

$$x' = x,\ y' = y/x; \qquad x = x',\ y = x'y',$$

having at $x = y = 0$ a fundamental point, and we denote by $\mathfrak{O}'$ the ring of polynomials in x', y'. $\mathfrak{O}$ is a subring of $\mathfrak{O}'$, and moreover $\mathfrak{O}'$ is contained in the valuation ring of the given valuation B, since, we have assumed $v(y) > v(x)$, whence $v(y') > 0$. Let $\{\mathfrak{q}'_j\}$ be the sequence of 0-dimensional v-ideals in $\mathfrak{O}'$, where $\mathfrak{q}'_1 = \mathfrak{p}' = (x', y')$. We wish to study the connection between the v-ideals $\mathfrak{q}_i$ in $\mathfrak{O}$ and the v-ideals $\mathfrak{q}'_j$ in $\mathfrak{O}'$. Note that $\mathfrak{q}_1 = \mathfrak{p} = (x, y)$, $\mathfrak{q}_2 = (y, x^2)$, whence $\mathfrak{O}'\mathfrak{p} = (x', x'y') = (x')$, where $(x') = \mathfrak{O}'x'$, and $\mathfrak{O}'\mathfrak{q}_2 = (x'y', x'^2) = x'\mathfrak{p}'$.

THEOREM 4.1. *The extended ideal of an ideal $\mathfrak{q}$ in the sequence $\{\mathfrak{q}_i\}$ is of the form $x'^h\mathfrak{q}'$, where $\mathfrak{q}'$ is an ideal in the sequence $\{\mathfrak{q}'_j\}$ and where $\mathfrak{q} \equiv 0(\mathfrak{p}^h)$, $\mathfrak{q} \not\equiv 0(\mathfrak{p}^{h+1})$. Moreover, $\mathfrak{q}$ is the contracted ideal of $x'^h\mathfrak{q}'$.*

Proof. If $f(x, y) = f_\sigma(x, y) + f_{\sigma+1}(x, y) + \cdots$ is a polynomial in x, y and $f_\sigma(x, y)$ is its subform, then

$$f(x, y) = f(x', x'y) = x'^\sigma f_\sigma(1, y') + x'^{\sigma+1} f_{\sigma+1}(1, y') + \cdots,$$

whence, considering f as an element of $\mathfrak{O}'$, we have $f \equiv 0(x'^\sigma)$, $f \not\equiv 0(x'^{\sigma+1})$. By hypothesis, the subform of any polynomial belonging to $\mathfrak{q}$ is of degree $\geqq h$ and $\mathfrak{q}$ contains a polynomial whose subform is exactly of degree h. Hence $\mathfrak{O}'\mathfrak{q} = x'^h\mathfrak{q}'$, where $\mathfrak{q}' \not\equiv 0(x')$.

We show that $\mathfrak{q}'$ is a v-ideal. We have

$$(8) \qquad v(\mathfrak{q}') = v(\mathfrak{q}) - hv(x).$$

Let ω be an element of $\mathfrak{O}'$ such that $v(\omega) \geqq v(\mathfrak{q}')$, and let

$$\omega = F(x', y') = F(x, y/x) = \frac{G(x, y)}{x^\sigma},$$

where F and G are polynomials and $G(x, y) \equiv 0(\mathfrak{p}^\sigma)$ (since clearly the subform of G is of degree $\geqq \sigma$). Since $v(\omega) \geqq v(\mathfrak{q}')$ it follows, by (8), $(h - \sigma)v(x) + v(G) \geqq v(\mathfrak{q})$. Let k be a non-negative integer such that

$$h_1 = h - \sigma + k \geqq 0.$$

We will have then

$$(9) \qquad v(x^{h_1}G) \geqq v(x^k\mathfrak{q}) = v(\mathfrak{p}^k\mathfrak{q}).$$

Let $\mathfrak{p}^k\mathfrak{q} \sim \mathfrak{q}_m$, where $\mathfrak{q}_m$ belongs to the sequence $\{\mathfrak{q}_i\}$ of v-ideals in $\mathfrak{O}$. The inequality (9) implies $x^{h_1}G \equiv 0(\mathfrak{q}_m)$, and since $x^{h_1}G \equiv 0(\mathfrak{p}^{h_1+\sigma}) \equiv 0(\mathfrak{p}^{h+k})$, we have

$$x^{h_1}G \equiv 0([\mathfrak{q}_m, \mathfrak{p}^{h+k}]).$$

By Corollary 3.3 of the preceding section it follows that

$$x^{h_1}G \equiv 0(\mathfrak{q}\mathfrak{p}^k),$$

whence $x^{h_1}G$ is of the form

$$x^{h_1}G = \Sigma A_i(x, y) B_i(x, y),$$

where $A_i \equiv 0(\mathfrak{q})$, $B_i \equiv 0(\mathfrak{p}^k)$. But then B_i/x^k is a polynomial in x', y', and putting $B'_i = B_i/x^k$ we have

$$x^{h-\sigma}G = x^{h_1}G/x^k = \Sigma A_i B'_i \equiv 0(\mathfrak{D}'\mathfrak{q}) \equiv 0(x'^h\mathfrak{q}'),$$

whence $\omega = G/x^\sigma \equiv 0(\mathfrak{q}')$. We have thus proved that the ideal $\mathfrak{q}'$ enjoys the property which characterizes the v-ideals: if $v(\omega) \geqq v(\mathfrak{q}')$, then $\omega \equiv 0(\mathfrak{q}')$.

It remains to prove that $\mathfrak{q}'$ belongs to the sequence $\{\mathfrak{q}'_j\}$. But this is obvious, since $\mathfrak{q}'$ is necessarily a 0-dimensional ideal[6] or the unit ideal. That the contracted ideal $\bar{\mathfrak{q}}$ of $x'^h\mathfrak{q}'$ coincides with $\mathfrak{q}$ follows immediately from the fact that $v(\bar{\mathfrak{q}}) \geqq v(x'^h\mathfrak{q}') = v(\mathfrak{q})$, whence $\bar{\mathfrak{q}} \equiv 0(\mathfrak{q})$, while on the other hand we must have, of course, $\mathfrak{q} \equiv 0(\bar{\mathfrak{q}})$. The theorem is thus proved.

The next theorem is in a sense the converse of the preceding theorem.

THEOREM 4. 2. *For any ideal $\mathfrak{q}'$ in the sequence $\{\mathfrak{q}'_j\}$ there exists an integer h such that $x'^h\mathfrak{q}'$ is the extended ideal of an ideal $\mathfrak{q}$ belonging to the sequence $\{\mathfrak{q}_i\}$.*

Proof. If $\phi_1(x', y')$, $\phi_2(x', y'), \cdots, \phi_k(x', y')$ is a base of $\mathfrak{q}'$ and if we write these polynomials in the form of quotients: $\phi_i(x', y') = \psi_i(x, y)/x^m$, with a common denominator x^m, where $\psi_1, \psi_2, \cdots, \psi_k$ are polynomials in x, y, then we see that $x'^m\mathfrak{q}'$ is the extended ideal of the ideal $(\psi_1, \psi_2, \cdots, \psi_k)$ in $\mathfrak{D}$. Thus, there exist integers m such that $x'^m\mathfrak{q}'$ is an extended ideal of an ideal in $\mathfrak{D}$. This will be true for all sufficiently high integers m, since if $\mathfrak{D}'\bar{\mathfrak{q}} = x'^m\mathfrak{q}'$, then $\mathfrak{D}'\mathfrak{p}\bar{\mathfrak{q}} = x'^{m+1}\mathfrak{q}'$. Let h be the smallest possible value of m, and let $\mathfrak{q}$ be the contracted ideal of $x'^h\mathfrak{q}$:

$$\mathfrak{q} = [\mathfrak{D}, x'^h\mathfrak{q}'], \qquad \mathfrak{D}'\mathfrak{q} = x'^h\mathfrak{q}'. \tag{10}$$

Since the contracted ideal of (x'^h) is $\mathfrak{p}^h$, it follows $\mathfrak{q} \equiv 0(\mathfrak{p}^h)$. Moreover, since the primary ideal $\mathfrak{q}'$ belongs to the prime ideal $\mathfrak{p}' = (x', y')$, we have $x'^n \equiv 0(x'^h\mathfrak{q}')$, if n is sufficiently high. Passing to the contracted ideals we find $\mathfrak{p}^n \equiv 0(\mathfrak{q})$. Hence there exists an ideal $\mathfrak{q}_n$ in the sequence $\{\mathfrak{q}_i\}$ of v-ideals in $\mathfrak{D}$ such that $v(\mathfrak{q}_n) = v(\mathfrak{q})$. Let $\mathfrak{p}^\sigma$ be the highest power of $\mathfrak{p}$ which divides

[6] If n is a sufficiently high integer, then $x^{n+h} \equiv 0(\mathfrak{q})$, whence $x'^n \equiv 0(\mathfrak{q}')$. Let $f(x, y)$ be a polynomial in $\mathfrak{q}$ whose subform is $f_h(x, y)$, $f_h \neq 0$. If $f = f_h + f_{h+1} + \cdots$, then the polynomial $f_h(1, y') + x' f_{h+1}(1, y') + \cdots$ belongs to $\mathfrak{q}'$. It follows that also $[f_h(1, y')]^n$ belongs to $\mathfrak{q}'$, hence $(x'^n, [f_h(1, y')]^n) \equiv 0(\mathfrak{q}')$, and consequently $\mathfrak{q}'$ is 0-dimensional, or is the unit ideal, since $f_h(1, y') \neq 0$.

$\mathfrak{q}_n$, $\mathfrak{q}_n \equiv 0(\mathfrak{p}^\sigma)$, $\mathfrak{q}_n \not\equiv 0(\mathfrak{p}^{\sigma+1})$. We have $\mathfrak{q} \equiv 0(\mathfrak{p}^h)$ and $\mathfrak{q} \not\equiv 0(\mathfrak{p}^{h+1})$, consequently $\sigma \leqq h$, because $\mathfrak{q} \equiv 0(\mathfrak{q}_n)$. Let $\mathfrak{O}'\mathfrak{q}_n = x'^\sigma \mathfrak{q}'_j$, where $\mathfrak{q}'_j$, by the preceding theorem, is a v-ideal in $\mathfrak{O}'$. Since $\mathfrak{q}_n \sim \mathfrak{q}$, we have $v(x'^\sigma \mathfrak{q}'_j) = v(x'^h \mathfrak{q}')$. If $\sigma = h$, then $v(\mathfrak{q}'_j) = v(\mathfrak{q}')$, whence $\mathfrak{q}'_j = \mathfrak{q}'$, since both are v-ideals. In this case $\mathfrak{q}_n$ and $\mathfrak{q}'$ must coincide, since both are contracted ideals of $x'^h\mathfrak{q}'$, and the theorem is proved.

Assume $\sigma < h$. We have

$$\mathfrak{q} \equiv 0([\mathfrak{q}_n, \mathfrak{p}^h]). \tag{11}$$

If $f(x, y)$ is any polynomial belonging to $[\mathfrak{q}_n, \mathfrak{p}^h]$, then $f = x'^h f'(x'y')$ and $f \equiv 0(x'^\sigma \mathfrak{q}'_j)$. Since $v(x'^\sigma \mathfrak{q}'_j) = v(x'^h \mathfrak{q}')$, it follows that $f'(x', y') \equiv 0(\mathfrak{q}')$. Hence $f \equiv 0(x'^h \mathfrak{q}')$ and consequently, by (10), $f \equiv 0(\mathfrak{q})$. We have therefore $[\mathfrak{q}_n, \mathfrak{p}^h] \equiv 0(\mathfrak{q})$, and, by (11), it follows

$$\mathfrak{q} = [\mathfrak{q}_n, \mathfrak{p}^h]. \tag{12}$$

We have $\mathfrak{q}_n \equiv 0(\mathfrak{p}^\sigma)$, $\mathfrak{q}_n \not\equiv 0(\mathfrak{p}^{\sigma+1})$ and $h > \sigma$. We can then apply Theorem 3 and we obtain $\mathfrak{q} = \mathfrak{p}^r \mathfrak{q}_a$, where $\mathfrak{q}_a$ is again a v-ideal in $\mathfrak{O}$. Here r is the dimension of $\Omega(\mathfrak{q})$ and consequently $r > 0$, because $\mathfrak{p}^{h-\sigma}\mathfrak{q}_n \equiv 0(\mathfrak{q})$, whence $r \geqq h - \sigma > 0$. We have then

$$\mathfrak{O}'\mathfrak{q} = \mathfrak{O}'\mathfrak{p}^r\mathfrak{q}_a = x'^r\mathfrak{O}'\mathfrak{q}_a = x'^h\mathfrak{q}',$$

whence $\mathfrak{O}'\mathfrak{q}_a = x'^{h-r}\mathfrak{q}'$. This contradicts our hypothesis that h is the smallest integer such that $x'^h\mathfrak{q}'$ is an extended ideal. Hence $\sigma < h$ is impossible, and the theorem is proved.

If $\mathfrak{A}$ is any ideal in $\mathfrak{O}$ and if $\mathfrak{O}'\mathfrak{A} = x'^h\mathfrak{A}'$, $\mathfrak{A}' \not\equiv 0(x')$, we shall call $\mathfrak{A}'$ the *transformed ideal* of $\mathfrak{A}$ (under the quadratic transformation T), in symbols: $\mathfrak{A}' = T(\mathfrak{A})$. By Theorems 4.1 and 4.2, the transform of any v-ideal $\mathfrak{q}$ in $\mathfrak{O}$ is a v-ideal $\mathfrak{q}'$ in $\mathfrak{O}'$, and every v-ideal $\mathfrak{q}'$ in $\mathfrak{O}'$ is the transform of at least one v-ideal $\mathfrak{q}$ in $\mathfrak{O}$, everything referred to a fixed valuation. There may be more than one v-ideal in $\mathfrak{O}$ whose transform is $\mathfrak{q}'$. To find them, we again consider the smallest integer h such that $x'^h\mathfrak{q}'$ is an extended ideal. It follows from the preceding proof that if $\mathfrak{q}$ is the contracted ideal of $x'^h\mathfrak{q}'$, then $T(\mathfrak{q}) = \mathfrak{q}'$, and from Theorem 4.1 it follows, that any other v-ideal $\bar{\mathfrak{q}}$ in $\mathfrak{O}$ such that $T(\bar{\mathfrak{q}}) = \mathfrak{q}'$ must be the contracted ideal of $x'^\sigma\mathfrak{q}'$, where σ is some integer greater than h. For a given integer σ greater than h, the contracted ideal of $x'^\sigma\mathfrak{q}'$ may or may not be a v-ideal in $\mathfrak{O}$, but let us at any rate examine this contracted ideal. Let us denote it by $\mathfrak{A}_\sigma$. Since $x'^\sigma\mathfrak{q}'$ is the extended ideal of $\mathfrak{A}_\sigma$ and *also* of $\mathfrak{p}^{\sigma-h}\mathfrak{q}$, it follows

$$\mathfrak{p}^{\sigma-h}\mathfrak{q} \equiv 0(\mathfrak{A}_\sigma). \tag{13}$$

For the same reason we have $v(\mathfrak{A}_\sigma) = v(\mathfrak{p}^{\sigma-h}\mathfrak{q})$. Let $\mathfrak{q}_m$ be the v-ideal equivalent to both $\mathfrak{A}_\sigma$ and $\mathfrak{p}^{\sigma-h}\mathfrak{q}$, $\mathfrak{q}_m \sim \mathfrak{A} \sim \mathfrak{p}^{\sigma-h}\mathfrak{q}$. By Theorem 3, Corollary 3. 3, we have $\mathfrak{p}^{\sigma-h}\mathfrak{q} = [\mathfrak{q}_m, \mathfrak{p}^\sigma]$. Now, $\mathfrak{A}_\sigma \sim \mathfrak{q}_m$ implies $\mathfrak{A}_\sigma \equiv 0(\mathfrak{q}_m)$ and since $\mathfrak{O}'\mathfrak{A}_\sigma = x'^\sigma\mathfrak{q}'$ we also have $\mathfrak{A}_\sigma \equiv 0(\mathfrak{p}^\sigma)$. Consequently $\mathfrak{A}_\sigma \equiv 0(\mathfrak{p}^{\sigma-h}\mathfrak{q})$, and hence, by (13), $\mathfrak{A}_\sigma = \mathfrak{p}^{\sigma-h}\mathfrak{q}$. We therefore can state the following theorem.

THEOREM 4. 3. *If $\mathfrak{q}'$ is a v-ideal in $\mathfrak{O}'$, belonging to the sequence $\{\mathfrak{q}'_j\}$, and if h is the smallest integer such that $x'^h\mathfrak{q}'$ is an extended ideal of an ideal in $\mathfrak{O}$, then the contracted ideal $\mathfrak{q}$ of $x'^h\mathfrak{q}'$ is a v-ideal in $\mathfrak{O}$, a member of the sequence $\{\mathfrak{q}_i\}$, and $T(\mathfrak{q}) = \mathfrak{q}'$. If σ is any integer $\geqq h$, the contracted ideal of $x'^\sigma\mathfrak{q}'$ is $\mathfrak{p}^{\sigma-h}\mathfrak{q}$, but need not be a v-ideal. In particular, the v-ideals $\mathfrak{q}_i$ whose transform is the given v-ideal $\mathfrak{q}'$ are all of the form $\mathfrak{p}^{\sigma-h}\mathfrak{q}$, $\sigma \geqq h$.*

We shall regard $\mathfrak{q}$ as the transform of $\mathfrak{q}'$ by T^{-1}: $\mathfrak{q} = T^{-1}(\mathfrak{q}')$. By the definition of $\mathfrak{q}$, the system $\Omega(\mathfrak{q})$ of the subforms of $\mathfrak{q}$ must be of dimension $r = 0$, i. e. every form in $\Omega(\mathfrak{q})$ differs from y^h by a factor c, $c \varepsilon \mathfrak{k}$. In fact, if it were $r > 0$, then $\mathfrak{q}$ could be put in the form $\mathfrak{q} = \mathfrak{p}^r\tilde{\mathfrak{q}}$, where $\tilde{\mathfrak{q}}$ is also a v-ideal (Corollary 3. 2), and we would have $\mathfrak{O}'\tilde{\mathfrak{q}} = x'^{h-r}\mathfrak{q}'$, contrary to our assumption that h is the smallest integer such that $x'^h\mathfrak{q}'$ is an extended ideal. Conversely, if $\mathfrak{q}$ is a v-ideal in $\mathfrak{O}$, belonging to the sequence $\{\mathfrak{q}_i\}$, and if $\Omega(\mathfrak{q})$ is of dimension zero, then $\mathfrak{q}$ is so related to its transform $\mathfrak{q}' = T(\mathfrak{q})$, that $\mathfrak{q} = T^{-1}\mathfrak{q}'$, i. e. if $\mathfrak{O}'\mathfrak{q} = x'^h\mathfrak{q}'$, then h is the smallest integer such that $x'^h\mathfrak{q}'$ is an extended ideal. In fact, in the contrary case, $\mathfrak{q}$ could be put in the form $\mathfrak{p}^r\tilde{\mathfrak{q}}$, $r > 0$, (by Theorem 4. 3), contrary to the hypothesis that $\Omega(\mathfrak{q})$ is of dimension zero. Thus, there is a one to one correspondence between the ideals $\mathfrak{q}'$ of the sequence $\{\mathfrak{q}'_j\}$ and those ideals $\mathfrak{q}$ of the sequence $\{\mathfrak{q}_i\}$ whose system $\Omega(\mathfrak{q})$ of subforms is of dimension zero: to each $\mathfrak{q}'$ there corresponds a unique ideal $\mathfrak{q} = T^{-1}(\mathfrak{q}')$, and $\mathfrak{q}' = T(\mathfrak{q})$.

Let $\mathfrak{q}'_\alpha$, $\mathfrak{q}'_\beta$ be two distinct v-ideals in $\mathfrak{O}'$, and let $\mathfrak{q}_i = T^{-1}(\mathfrak{q}'_\alpha)$, $\mathfrak{q}_j = T^{-1}(\mathfrak{q}'_\beta)$, be their transforms in $\mathfrak{O}$. Suppose that $i < j$, whence $\mathfrak{q}_j \equiv 0(\mathfrak{q}_i)$. *We assert that in such a case also $\alpha < \beta$.* In fact, let $\mathfrak{q}_i \equiv 0(\mathfrak{p}^h)$, $\mathfrak{q}_i \not\equiv 0(\mathfrak{p}^{h+1})$ and let $\mathfrak{q}_j \equiv 0(\mathfrak{p}^\sigma)$, $\mathfrak{q}_j \not\equiv 0(\mathfrak{p}^{\sigma+1})$, whence $\mathfrak{O}'\mathfrak{q}_i = x'^h\mathfrak{q}'_\alpha$, $\mathfrak{O}'\mathfrak{q}_j = x'^\sigma\mathfrak{q}'_\beta$, and clearly, $\sigma \geqq h$. Evidently $\mathfrak{p}^{\sigma-h}\mathfrak{q}_i \equiv 0(x'^\sigma\mathfrak{q}'_\alpha)$. Supposing that $\alpha > \beta$, whence $\mathfrak{q}'_\alpha \equiv 0(\mathfrak{q}'_\beta)$, we would have $x'^\sigma\mathfrak{q}'_\alpha \equiv 0(x'^\sigma\mathfrak{q}'_\beta)$, and passing to the contracted ideals in $\mathfrak{O}$, we would get by Theorem 4. 3 $\mathfrak{p}^{\sigma-h}\mathfrak{q}_i \equiv 0(\mathfrak{q}_j)$. The equality $\sigma = h$ is excluded, because $\mathfrak{q}_j \equiv 0(\mathfrak{q}_i)$ and $\mathfrak{q}_j \neq \mathfrak{q}_i$. Hence $\sigma > h$, but then the congruence $\mathfrak{p}^{\sigma-h}\mathfrak{q}_i \equiv 0(\mathfrak{q}_j)$ is in contradiction with the fact that the $\Omega(\mathfrak{q}_j)$ (consisting of form of degree σ) is of dimension zero. Hence our assumption $\alpha > \beta$ leads to a contradiction, and consequently it is proved that $i < j$ implies $\alpha < \beta$. If then $T^{-1}(\mathfrak{q}'_j) = \mathfrak{q}_{\alpha_j}$, the indices α_j form an ascending

sequence, $\alpha_0 < \alpha_1 < \alpha_2 < \cdots$. It is immediately verified that $\alpha_0 = 0$, $\alpha_1 = 2$. Hence $\alpha_j > j$, if $j > 0$. Let now $\mathfrak{q}_s$ be any v-ideal in the sequence $\{\mathfrak{q}_i\}$ and let $T(\mathfrak{q}_s) = \mathfrak{q}'_\sigma$. By Theorem 4.3 we have $\mathfrak{q}_s = \mathfrak{p}^\rho \mathfrak{q}_{\alpha_\sigma}$, $\rho \geqq 0$, whence $\mathfrak{q}_s \equiv 0(q_{\alpha_\sigma})$, and $s \geqq \alpha_\sigma$, i. e. $s > \sigma$. Observing that the length of the ideal $\mathfrak{q}_s$ is equal to s, we can reassume the preceding results in the following theorem:

THEOREM 4.4. *If* $\mathfrak{q}_{\alpha_j} = T^{-1}(\mathfrak{q}'_j)$, *then* $\alpha_j > j$, *if* $j \geqq 1$. *Moreover, if* $\mathfrak{q}_s$ *is any ideal in the sequence* $\{\mathfrak{q}_i\}$ *and if* $\mathfrak{q}'_\sigma = T(\mathfrak{q}_s)$, *then* $s > \sigma$, *i. e. length of* $\mathfrak{q}_s >$ *length of* $\mathfrak{q}_\sigma$.

5. Simple and composite *v*-ideals. We say that an ideal $\mathfrak{A}$ in $\mathfrak{O}$ is *simple*, if $\mathfrak{A}$ cannot be represented as the product of two ideals, both different from the unit ideal, i. e. if $\mathfrak{A} = \mathfrak{B}\mathfrak{C}$, $\mathfrak{B} \neq (1)$ implies $\mathfrak{C} = (1)$. An ideal is *composite* if it is not simple.

THEOREM 5.1. *A composite v-ideal can be represented as a product of v-ideals different from the unit ideal.*

Proof. Let $\mathfrak{A}$ be a v-ideal in $\mathfrak{O}$, belonging to some valuation B, and let $\mathfrak{A} = \mathfrak{B}\mathfrak{C}$, $\mathfrak{B} \neq (1)$, $\mathfrak{C} \neq (1)$. Let $\mathfrak{B}_1$, $\mathfrak{C}_1$ be the v-ideals belonging to B such that $\mathfrak{B} \sim \mathfrak{B}_1$, $\mathfrak{C} \sim \mathfrak{C}_1$. Since $\mathfrak{B} \equiv 0(\mathfrak{B}_1)$, $\mathfrak{C} \equiv 0(\mathfrak{C}_1)$, we have $\mathfrak{A} \equiv 0(\mathfrak{B}_1\mathfrak{C}_1)$. On the other hand, $v(\mathfrak{A}) = v(\mathfrak{B}) + v(\mathfrak{C}) = v(\mathfrak{B}_1) + v(\mathfrak{C}_1) = v(\mathfrak{B}_1\mathfrak{C}_1)$, whence $\mathfrak{B}_1\mathfrak{C}_1 \equiv 0(\mathfrak{A})$, since $\mathfrak{A}$ is a v-ideal. We conclude that $\mathfrak{A} = \mathfrak{B}_1\mathfrak{C}_1$, and it remains to prove that $\mathfrak{B}_1 \neq (1)$ and $\mathfrak{C}_1 \neq (1)$. Assume the contrary, and let, for instance $\mathfrak{B}_1 = (1)$. Then $\mathfrak{A} = \mathfrak{C}_1$, whence $\mathfrak{C} \equiv 0(\mathfrak{A})$, and consequently $\mathfrak{C} = \mathfrak{A}$ since $\mathfrak{A} = \mathfrak{B}\mathfrak{C} \equiv 0(\mathfrak{C})$. We have then $\mathfrak{A} = \mathfrak{B}\mathfrak{A}$, and this implies ([2], p. 36), $\mathfrak{B} = (1)$, contrary to hypothesis.

Consider the given valuation B and the corresponding sequences $\{\mathfrak{q}_i\}$, $\{\mathfrak{q}'_j\}$ of 0-dimensional v-ideals in the polynomial rings $\mathfrak{O} = \mathfrak{k}[x,y]$, $\mathfrak{O}' = \mathfrak{k}[x',y']$ respectively. Let, as before, $T^{-1}(\mathfrak{q}'_j) = \mathfrak{q}_{\alpha_j}$. It is clear that all the simple v-ideals of the sequence $\{\mathfrak{q}_i\}$, except the ideal $\mathfrak{p} = \mathfrak{q}_1$, belong to the sequence $\{\mathfrak{q}_{\alpha_j}\}$, since any ideal $\mathfrak{q}_i$ $(i \neq 1)$, not in the sequence $\{\mathfrak{q}_{\alpha_j}\}$, is, by Theorem 4.3, either of the form $\mathfrak{p}^\rho \mathfrak{q}_{\alpha_j}$, $\rho > 0$, $\mathfrak{q}_{\alpha_j} \neq (1)$, or of the form $\mathfrak{p}^\rho$, $\rho > 1$. Now suppose that $\mathfrak{q}_{\alpha_j}$, for a given j, is a composite ideal. Then we can write, by Theorem 5.1, $\mathfrak{q}_{\alpha_j} = \mathfrak{q}_s \mathfrak{q}_t$, where $\mathfrak{q}_s$, $\mathfrak{q}_t$ are in the sequence $\{\mathfrak{q}_i\}$. Hence $\mathfrak{q}'_j = T(\mathfrak{q}_{\alpha_j}) = \mathfrak{q}'_\sigma \mathfrak{q}'_\tau$, where $\mathfrak{q}'_\sigma = T(\mathfrak{q}_s)$ and $\mathfrak{q}'_\tau = T(\mathfrak{q}_t)$. We have seen above that $\Omega(\mathfrak{q}_{\alpha_j})$ is of dimension zero; therefore neither $\mathfrak{q}_s$ nor $\mathfrak{q}_t$ can be a power of $\mathfrak{p}$. It follows that $\mathfrak{q}'_\sigma \neq (1)$ and $\mathfrak{q}'_\tau \neq (1)$, i. e. $\mathfrak{q}'_j$ is composite. We conclude then that if $\mathfrak{q}'_j$ is a simple v-ideal in $\mathfrak{O}'$, then its transform $\mathfrak{q}_{\alpha_j} = T^{-1}(\mathfrak{q}'_j)$ is also a simple v-ideal. Much more difficult is to prove the converse:

THEOREM 5.2. *The transform $T(\mathfrak{q}_i)$ of a simple v-ideal $\mathfrak{q}_i$ in $\mathfrak{O}$ is a simple v-ideal (in $\mathfrak{O}'$).*

The proof of this theorem will be given in Part II of the paper (Corollary 11.2), where we shall characterize the simple v-ideals from the point of view of formal power series. Here we shall use this theorem without proof.

Let $\mathfrak{P}_1, \mathfrak{P}_2, \cdots, \mathfrak{P}_i, \cdots$ be the sequence of simple v-ideals, different from (1), as they occur in the sequence $\{\mathfrak{q}_i\}$:

$$(14) \qquad \mathfrak{P}_1 \supset \mathfrak{P}_2 \supset \cdots, \qquad \mathfrak{P}_1 = \mathfrak{p} = (x, y),\ \mathfrak{P}_2 = \mathfrak{q}_2 = (y, x^2).$$

Let similarly $\mathfrak{P}'_1, \mathfrak{P}'_2, \cdots$ be the sequence of simple v-ideals, different from (1), as they occur in the sequence $\{\mathfrak{q}'_\nu\}$. By the preceding results, especially by Theorem 5.2, there is (1, 1) correspondence between the ideals $\mathfrak{P}_i$, $i > 1$, and the ideals $\mathfrak{P}'_\alpha$, where to $\mathfrak{P}_i$ corresponds $T(\mathfrak{P}_i) = \mathfrak{P}'_\alpha$, and $\mathfrak{P}_i = T^{-1}(\mathfrak{P}'_\alpha)$. Moreover, by Theorem 4.4, if $T(\mathfrak{P}_i) = \mathfrak{P}'_\alpha$, and $T(\mathfrak{P}_j) = \mathfrak{P}'_\beta$, then $i < j$ implies $\alpha < \beta$. Since $T(\mathfrak{P}_2) = \mathfrak{P}'_1$, we conclude with the following theorem:

THEOREM 5.3. *$T(\mathfrak{P}_i) = \mathfrak{P}'_{i-1}$, i. e. the transform of the simple v-ideal $\mathfrak{P}_i$ by the quadratic transformation T is the simple v-ideal $\mathfrak{P}'_{i-1}$.*

As a consequence of this theorem, it follows incidentally that the sequence $\{\mathfrak{q}_n\}$ contains infinitely many simple v-ideals. In fact, if we assume that any sequence $\{\mathfrak{q}_n\}$ of v-ideals in any polynomial ring contains always at least $k > 0$ simple v-ideals (it always contains at least one, namely the prime ideal $\mathfrak{q}_1 = \mathfrak{p} = (x, y)$), and if we apply this assumption to the sequence $\{\mathfrak{q}'_\nu\}$ of v-ideals in the polynomial ring $\mathfrak{O}'$, we deduce immediately, by Theorem 5.3, that any sequence $\{\mathfrak{q}_n\}$ contains at least $k + 1$ simple v-ideals.

6. Properties of simple *v*-ideals. Heretofore we have been dealing with a fixed valuation B and with the v-ideals in $\mathfrak{O}$ belonging to B. Now, a v-ideal belonging to B may also occur as a v-ideal for many other valuations. Consider, in particular, the i-th simple v-ideal $\mathfrak{P}_i$ for B, and let $\bar{B}$ be another valuation for which $\mathfrak{P}_i$ is a v-ideal. We assert that $\mathfrak{P}_i$ *is also the i-th simple v-ideal for $\bar{B}$.* The assertion is trivial for $i = 1$, because $\mathfrak{P}_1 = \mathfrak{p} = (x, y)$. We may then proceed by induction, assuming that our assertion is true for $i - 1$. Let $\{\mathfrak{q}_n\}$ and $\{\bar{\mathfrak{q}}_n\}$ be the sequences of v-ideals in $\mathfrak{O}$ for the valuations B and $\bar{B}$ respectively. Since $\mathfrak{P}_i$ occurs in both sequences, and since the ideals $\mathfrak{q}_n$ and $\bar{\mathfrak{q}}_n$ are primary ideals belonging to the prime 0-dimensional ideals $\mathfrak{q}_1$ and $\bar{\mathfrak{q}}_1$ respectively, it follows that $\mathfrak{q}_1 = \bar{\mathfrak{q}}_1 = \mathfrak{p} = (x, y)$. Furthermore, since $\mathfrak{P}_i$ is simple, its system $\Omega(\mathfrak{P}_i)$ of subforms is of dimension 0. If $cx + dy$ is the base of $\Omega(\mathfrak{P}_i)$, we must have $v(cx + dy) > v(\mathfrak{p})$ in B and also

$v(cx+dy) > v(\mathfrak{p})$ in $\bar{B}$. We may therefore assume that $v(y) > v(x)$ in both valuations B and $\bar{B}$. We apply the quadratic transformation $T: x' = x,\ y' = y/x$ to both sequences $\{\mathfrak{q}_n\}$ and $\{\bar{\mathfrak{q}}_n\}$. The sequences $\{\mathfrak{q}'_\nu\}$ and $\{\bar{\mathfrak{q}}'_\nu\}$ of v-ideals in $\mathfrak{O}' = \mathfrak{k}[x', y']$ belonging to the valuations B and $\bar{B}$ respectively, will consist of primary ideals belonging to the prime ideal $\mathfrak{p}' = (x', y')$. The transform $\mathfrak{P}'_{i-1}$ of $\mathfrak{P}_i$ belongs to both sequences and is the $(i-1)$-th simple v-ideal in the sequence $\{\mathfrak{q}'_\nu\}$. Hence, by our induction, $\mathfrak{P}'_{i-1}$ is also the $(i-1)$-th simple ideal in the sequence $\{\bar{\mathfrak{q}}'_\nu\}$. As a consequence, $\mathfrak{P}_i$ must be the i-th simple ideal in the sequence $\{\bar{\mathfrak{q}}_n\}$, and this proves our assertion.

Thus, given a simple v-ideal $\mathfrak{P} = \mathfrak{P}_i$ in $\mathfrak{O}$, there is uniquely determined an integer i, such that $\mathfrak{P}$ is the i-th simple v-ideal in the sequence of simple v-ideals of any valuation for which $\mathfrak{P}$ is a valuation ideal. We shall say that $\mathfrak{P}$ is *a simple v-ideal of kind i*, by analogy with the terminology of the geometric theory of infinitely near points in the plane, where a point $O^{(i)}$, infinitely near the point $O^{(1)} \equiv (0,0)$, is said to be of kind i, if it is in the $(i-1)$-th neighborhood of $O^{(1)}$. The identity of the two concepts will appear from the formal power series considerations of Part II. However, already at this stage, the analogy appears from the fact, that while it takes $i-1$ successive quadratic transformations to transform a point $O^{(i)}$ of kind i into a proper point (a point of kind 1), it takes as well $i-1$ successive quadratic transformations to transform a simple v-ideal $\mathfrak{P}_i$ of kind i into a simple v-ideal of kind 1, i. e. into a prime 0-dimensional ideal.

THEOREM 6.1. *If $\{\mathfrak{P}_\alpha\}$ and $\{\bar{\mathfrak{P}}_\alpha\}$ are the sequences of simple v-ideals in $\mathfrak{O}$ belonging to valuations B and $\bar{B}$ respectively and if, for a given i, we have $\mathfrak{P}_i = \bar{\mathfrak{P}}_i$, then also $\mathfrak{P}_\alpha = \bar{\mathfrak{P}}_\alpha$ for any $\alpha < i$. In other words, the $i-1$ simple v-ideals which precede a given simple v-ideal $\mathfrak{P}_i$ of kind i in a given valuation B for which $\mathfrak{P}_i$ is a v-ideal, are uniquely determined by $\mathfrak{P}_i$, being independent of the valuation B.*

Proof by induction. The theorem is trivial for $i = 1$. Assume that the theorem is true for simple v-ideals of kind $i-1$, and apply the quadratic transformation T. We will have then in $\mathfrak{O}'$: $\mathfrak{P}'_{i-1} = \bar{\mathfrak{P}}'_{i-1}$, where $\mathfrak{P}'_{\alpha-1} = T(\mathfrak{P}_\alpha)$ and $\bar{\mathfrak{P}}'_{\alpha-1} = T(\bar{\mathfrak{P}}_\alpha)$. Hence, by our induction, $\mathfrak{P}'_\alpha = \bar{\mathfrak{P}}'_\alpha$, for $\alpha = 1, 2, \cdots, i-2$, whence $\mathfrak{P}_\alpha = \bar{\mathfrak{P}}_\alpha$ for $\alpha = 2, \cdots, i-1$, because $\mathfrak{P}_\alpha$ as well as $\bar{\mathfrak{P}}_\alpha$ is the contracted ideal of the ideal $x'^h\mathfrak{P}'_{\alpha-1}$ $(= x'^h\bar{\mathfrak{P}}'_{\alpha-1})$, where h is the smallest integer such that $x'^h\mathfrak{P}'_{\alpha-1}$ is an extended ideal of an ideal in $\mathfrak{O}$. Moreover, $\mathfrak{P}_1 = \mathfrak{P}'_1$, since $\mathfrak{P}_1$ and $\mathfrak{P}'_1$ are the prime ideals belonging to $\mathfrak{P}_i$ and $\mathfrak{P}'_i$ respectively q. e. d.

A much stronger theorem can be proved:

THEOREM 6.2. *Under the hypothesis $\mathfrak{P}_i = \bar{\mathfrak{P}}_i$ of Theorem* 6.1, *the set of v-ideals for B which precede $\mathfrak{P}_i$ (divisors of $\mathfrak{P}_i$) coincides with the set of v-ideals for $\bar{B}$ which precede $\bar{\mathfrak{P}}_i$.*

Proof by induction with respect to i. For $i = 1$ the theorem is trivial. Assume that the theorem is true for simple v-ideals of kind $i - 1$. Let $\mathfrak{q}_k$ be a v-ideal for B such that $\mathfrak{P}_i \equiv 0(\mathfrak{q}_k)$. If $\mathfrak{q}_k$ is simple, then it is also a v-ideal for $\bar{B}$, by the preceding theorem. If $\mathfrak{q}_k$ is composite, we know by Theorem 5.1 that it can be factored into simple v-ideals belonging to B. Since $\mathfrak{q}_k$ is a proper divisor of $\mathfrak{P}_i$, only factor $\mathfrak{P}_j$, $j < i$, can occur. Let

$$\mathfrak{q}_k = \mathfrak{P}_1^{\alpha_1}\mathfrak{P}_2^{\alpha_2}\cdots\mathfrak{P}_{i-1}^{\alpha_{i-1}}, \qquad \mathfrak{P}_1 = \mathfrak{p} = (x, y).$$

We consider separately two cases: (1) $\alpha_1 = 0$, (2) $\alpha_1 > 0$.

(1) First case: $\alpha_1 = 0$. We apply our quadratic transformation T. We find then

$$T(\mathfrak{q}_k) = \mathfrak{q}'_\sigma = \mathfrak{P}'^{\alpha_2}_1 \cdots \mathfrak{P}'^{\alpha_{i-1}}_{i-2}. \tag{15}$$

Since the factor $\mathfrak{P}_1 (= \mathfrak{p})$ does not occur in the factorization of $\mathfrak{q}_k$, the system $\Omega(\mathfrak{q}_k)$ of subforms of $\mathfrak{q}_k$ is of dimension zero. Hence $\mathfrak{q}_k = T^{-1}(\mathfrak{q}'_\sigma)$. Since also $\mathfrak{P}_i = T^{-1}(\mathfrak{P}'_{i-1})$ and since $\mathfrak{P}_i \equiv 0(\mathfrak{q}_k)$, we have, by Theorem 4.4, $\mathfrak{P}'_{i-1} \equiv 0(\mathfrak{q}'_\sigma)$. Now $\mathfrak{P}'_{i-1}$ is a v-ideal in $\mathfrak{O}'$ for both B and $\bar{B}$. Hence, by our induction, $\mathfrak{q}'_\sigma$ is also a v-ideal for $\bar{B}$. But then also $\mathfrak{q}_k$ must be a v-ideal in $\mathfrak{O}$ for $\bar{B}$, since $\mathfrak{q}_k = T^{-1}(\mathfrak{q}'_\sigma)$.

(2) Second case: $\alpha_1 > 0$. We now use an induction with respect to k, i.e. we assume it has been already proved that all the v-ideals $\mathfrak{q}_j$ for B, $j < k$, are also v-ideals for $\bar{B}$. Since in the factorization (15) the factor $\mathfrak{p}$ occurs to the power α_1, it follows that the system $\Omega(\mathfrak{q}_k)$ of subforms of $\mathfrak{q}_k$ is of dimension α_1. Hence, by Theorem 3 (Corollary 3.2) we can write $\mathfrak{q}_k = \mathfrak{p}^{\alpha_1}\mathfrak{q}_t$, where $\mathfrak{q}_t$ is a v-ideal for B. Let $\mathfrak{p}^{\alpha_1 - 1}\mathfrak{q}_t \sim \mathfrak{q}_s$. We have $\mathfrak{q}_k \equiv 0(\mathfrak{p}\mathfrak{q}_s)$; on the other hand, since $v(\mathfrak{p}^{\alpha_1-1}\mathfrak{q}_t) = v(\mathfrak{q}_s)$, it follows that $v(\mathfrak{p}\mathfrak{q}_s) = v(\mathfrak{p}^{\alpha_1}\mathfrak{q}_t) = v(\mathfrak{q}_k)$, whence $\mathfrak{p}\mathfrak{q}_s \equiv 0(\mathfrak{q}_k)$. Consequently $\mathfrak{q}_k = \mathfrak{p}\mathfrak{q}_s$. We now consider the k-th v-ideal $\bar{\mathfrak{q}}_k$ for $\bar{B}$. We must suppose that the factor $\mathfrak{P}_1$ occurs also in the factorization of $\bar{\mathfrak{q}}_k$, since otherwise $\bar{\mathfrak{q}}_k$ would also be a v-ideal for B, by the preceding case $\alpha_1 = 0$, whence necessarily $\bar{\mathfrak{q}}_k = \mathfrak{q}_k$, since k is the length of both ideals $\mathfrak{q}_k, \bar{\mathfrak{q}}_k$. Let then $\bar{\mathfrak{q}}_k = \mathfrak{p}\bar{\mathfrak{q}}_\sigma$. By our induction, $\bar{\mathfrak{q}}_\sigma$ is a v-ideal for B, since $\sigma < k$; i.e. $\bar{\mathfrak{q}}_\sigma = \mathfrak{q}_\sigma$. Hence $\mathfrak{q}_k = \mathfrak{p}\mathfrak{q}_s$, $\bar{\mathfrak{q}}_k = \mathfrak{p}\mathfrak{q}_\sigma$. Since of the two ideals $\mathfrak{q}_s$, $\mathfrak{q}_\sigma$ one is a divisor of the other, the same is true of the ideals $\mathfrak{q}_k$ and $\bar{\mathfrak{q}}_k$. But these two ideals have the same length k, consequently $\mathfrak{q}_k = \bar{\mathfrak{q}}_k$, q. e. d.

Remark. The following example shows that a composite v-ideal does not determine the v-ideals preceding it. Let B be the valuation defined by the branch $y = x^{3/2}$ and let $\bar{B}$ be the valuation determined by the branch $y = x^{2/3}$. We have then $\mathfrak{q}_1 = \mathfrak{p} = (x, y)$, $\mathfrak{q}_2 = (y, x^2)$, $\mathfrak{q}_3 = \mathfrak{p}^2$, while $\bar{\mathfrak{q}}_1 = \mathfrak{p} = \mathfrak{q}_1$, $\bar{\mathfrak{q}}_2 = (x, y^2) \neq \mathfrak{q}_2$, $\bar{\mathfrak{q}}_3 = \mathfrak{p}^2 = \mathfrak{q}_3$.

7. A factorization theorem for v-ideals. We know from the preceding sections that any v-ideal $\mathfrak{A}$, belonging to a given valuation B, can be factored into simple v-ideals belonging to the same valuation B. The question arises as to the *unicity of this factorization.* The unicity of the factorization may be *a priori* intended in more than one way. In the first place, we *may fix some valuation B* to which $\mathfrak{A}$ belongs, and we may ask whether the factorization of $\mathfrak{A}$ into simple v-ideals *belonging to B* is unique. We may go a step further and ask whether the factorization of $\mathfrak{A}$, if unique for a given valuation B, is *independent of B.* Finally, we may formulate the unicity of factorization of $\mathfrak{A}$ in its strongest possible form and assert, that $\mathfrak{A}$ *can be factored in a unique manner into simple v-ideals,* where we allow *a priori* that the simple v-factors may belong to different valuations. It is this strongest form of the unicity theorem which we proceed to prove. It will, of course, follow from this theorem, that the simple v-factors are v-ideals for any valuation for which $\mathfrak{A}$ is a v-ideal.

We prove, however, a stronger theorem, from which the unique factorization of v-ideals into simple v-ideals will follow:

THEOREM 7.1. *Let $\mathfrak{A}_1, \mathfrak{A}_2, \cdots, \mathfrak{A}_k$ and $\bar{\mathfrak{A}}_1, \bar{\mathfrak{A}}_2, \cdots, \bar{\mathfrak{A}}_s$ be two sets of simple v-ideals belonging to valuations $B_1, B_2, \cdots, B_k$ and $\bar{B}_1, \bar{B}_2, \cdots, \bar{B}_s$ respectively. If*

$$\mathfrak{A}_1^{\alpha_1}\mathfrak{A}_2^{\alpha_2} \cdots \mathfrak{A}_k^{\alpha_k} = \bar{\mathfrak{A}}_1^{\bar{\alpha}_1}\bar{\mathfrak{A}}_2^{\bar{\alpha}_2} \cdots \bar{\mathfrak{A}}_s^{\bar{\alpha}_s}, \tag{16}$$

where the α_i's and $\bar{\alpha}_j$'s are positive integers, then necessarily $k = s$ and, for a proper arrangement of the indices, $\mathfrak{A}_i = \bar{\mathfrak{A}}_i$, $\alpha_i = \bar{\alpha}_i$.

This theorem is stronger, because a product of v-ideals, in particular the power product $\Pi\mathfrak{A}_i^{\alpha_i}$, is not necessarily a v-ideal.[7]

Proof. Let the simple v-ideals $\mathfrak{A}_i$ and $\bar{\mathfrak{A}}_j$ be of kind h_i and $\bar{h}_j$ respectively[8]; and let $m = \max.(h_1, \cdots, h_k, \bar{h}_1, \cdots, \bar{h}_s)$. The theorem is trivial

[7] For instance, let $\mathfrak{A}_1 = (y, x^2)$, $\mathfrak{A}_2 = (x, y^2)$. Both $\mathfrak{A}_1$, $\mathfrak{A}_2$ are v-ideals, but $\mathfrak{A}_1\mathfrak{A}_2 = (xy, \mathfrak{p}^3)$ is not a v-ideal, because the system $\Omega(\mathfrak{A}_1\mathfrak{A}_2)$ of subforms is of dimension zero and its base xy is not the power of a linear form (see Theorem 3).

[8] We assume that all the ideals $\mathfrak{A}_i$, $\mathfrak{A}_j$ are zero-dimensional (necessarily primary). The case of one-dimensional simple v-ideals is trivial, because any such ideal is prime,

in the case $m = 1$. In fact, in this case the ideals $\mathfrak{A}_i$ and $\bar{\mathfrak{A}}_j$ are prime zero-dimensional ideals, and the power products on both sides of (16) coincide with $[\mathfrak{A}_1^{\alpha_1}, \mathfrak{A}_2^{\alpha_2}, \cdots, \mathfrak{A}_k^{\alpha_k}]$ and $[\bar{\mathfrak{A}}_1^{\bar{\alpha}_1}, \bar{\mathfrak{A}}_2^{\bar{\alpha}_2}, \cdots, \bar{\mathfrak{A}}_s^{\bar{\alpha}_s}]$ respectively. The theorem follows in this case from the unicity of the decomposition of a zero-dimensional ideal into primary components. We may then prove our theorem by induction with respect to m.

The partial power products on both sides of (16) consisting of factors which belong to one and the same zero-dimensional ideal, must be equal to each other. Hence it is sufficient to prove the theorem for the case in which the ideals $\mathfrak{A}_i$, $\bar{\mathfrak{A}}_j$ all belong to one and the same prime ideal, say to $\mathfrak{p} = (x, y)$. The ideal $\mathfrak{p}$ consists then, for each of the given valuations B_i, $\bar{B}_j$, of all the polynomials whose value is > 0. We may assume that $v(x) = v(\mathfrak{p})$ in any of the valuations B_i, $\bar{B}_j$. Let us now apply the quadratic transformations $T: x' = x$, $y' = y/x$, and let $T(\mathfrak{A}_i) = \mathfrak{A}'_i$, $T(\bar{\mathfrak{A}}_j) = \bar{\mathfrak{A}}'_j$. If we have, in the valuation B_i, $v\left(\frac{y - c_i x}{x}\right) > 0$, then $\mathfrak{A}'_i$ will be a primary ideal in $\mathfrak{O}' = \mathfrak{k}[x', y']$ belonging to the ideal $(x', y' - c_i)$. A similar remark holds for $\bar{\mathfrak{A}}'_j$. At any rate, $\mathfrak{A}'_i$ will be a simple v-ideal of kind $h_i - 1$ and $\bar{\mathfrak{A}}'_j$ will be a simple v-ideal of kind $\bar{h}_j - 1$. We must remember, however, that the transform of a simple v-ideal of kind 1, i. e. of $\mathfrak{p}$, is the unit ideal $\mathfrak{O}'$. If then $\mathfrak{A}_1 = \bar{\mathfrak{A}}_1 = \mathfrak{p}$, where we allow now that one or both of the exponents α_1, $\bar{\alpha}_1$ may be zero, operating by T on (16) we get

$$\mathfrak{A}'^{\alpha_2}_2 \mathfrak{A}'^{\alpha_3}_3 \cdots \mathfrak{A}'^{\alpha_k}_k = \bar{\mathfrak{A}}'^{\bar{\alpha}_2}_2 \bar{\mathfrak{A}}'^{\bar{\alpha}_3}_3 \cdots \bar{\mathfrak{A}}'^{\bar{\alpha}_s}_s.$$

Since $\max.(h_i - 1, \bar{h}_j - 1) = m - 1$, we have by our induction, $k = s$, $\alpha_i = \bar{\alpha}_i$, $\mathfrak{A}'_i = \bar{\mathfrak{A}}'_i$ $(i > 1)$, and (16) becomes

$$\mathfrak{p}^{\alpha_1} \mathfrak{A}_2^{\alpha_2} \cdots \mathfrak{A}_k^{\alpha_k} = \mathfrak{p}^{\bar{\alpha}_1} \mathfrak{A}_2^{\alpha_2} \cdots \mathfrak{A}_k^{\alpha_k}.$$

Now α_1 is the dimension of the system $\Omega(\mathfrak{p}^{\alpha_1} \mathfrak{A}_2^{\alpha_2} \cdots \mathfrak{A}_k^{\alpha_k})$ and similarly $\bar{\alpha}_1$ is the dimension of $\Omega(\mathfrak{p}^{\bar{\alpha}_1} \mathfrak{A}_2^{\alpha_2} \cdots \mathfrak{A}_k^{\alpha_k})$. Hence $\alpha_1 = \bar{\alpha}_1$ and the theorem is proved.

PART II.

8. Algebraic and transcendental valuations. In this part of the paper we shall use the apparatus of formal power series in order to derive further

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and is therefore a principal ideal (f), where f is an irreducible polynomial. The one-dimensional factors and their exponents on both sides of (16) must be the same and may be deleted.

properties of valuation ideals in the ring of polynomials. The use of formal power series is clearly indicated by the fact that a zero-dimensional valuation of the field Σ of rational functions of x and y is essentially a local property of the field. It is known that any zero-dimensional valuation B of rank 1, in which x and y have positive values, can be obtained by the following construction: We put

$$x = A(t) = \alpha_1 t^{a_1} + \alpha_2 t^{a_2} + \cdots, \qquad y = B(t) = \beta_1 t^{b_1} + \beta_2 t^{b_2} + \cdots,$$

where the coefficients α and β belong to the underlying field $\mathfrak{f}$ and where the exponents a_i, b_i of each power series $A(t)$ and $Q(t)$ form a monotonic increasing sequence of positive real numbers. By substitution of these power series every element r of Σ takes a definite form

$$r = \gamma_1 t^{c_1} + \gamma_2 t^{c_2} \cdots (\gamma_1 \neq 0,\ c_1 \gtreqless 0),$$

and the valuation B is obtained by putting $v(r) = c_1$. We may eliminate formally t between $x = A(t)$ any $y = B(t)$ and we may thus define the valuation B by putting $y = P(x) = \delta_1 x^{d_1} + \delta_2 x^{d_2} + \cdots$, where the exponents are again increasing positive real numbers.

Now we may effect the substitution $x = A(t)$, $y = B(t)$ not only in any rational function r of x, y but also in any formal power series $\xi = \sum_{i,j} a_{ij} x^i y^j$, $i, j \geqq 0$, i, j-integers, and in any quotient of such formal power series. In this manner the valuation B defines a valuation B^* of the field Σ^* of meromorphic functions of x, y, and the valuation ring of B^* contains the ring $\mathfrak{D}^*$ of holomorphic functions ξ.

The special case in which the exponents a_i, b_i of the power series $A(t)$, $B(t)$ are integers, is the only one which is of interest in the classical theory of algebraic and analytic functions. The corresponding valuations B may be called *algebroid* or *analytic,* while non-analytic valuations may be referred to as *transcendental valuations.* For an algebroid valuation the power series $P(x)$ is an ordinary Puiseux series, i. e. the exponents $d_1, d_2, \cdots$ are rational numbers with fixed denominator:

$$d_i = m_i/n, \qquad (n, m_1, m_2, \cdots) = 1.$$

If the branch $y = P(x)$ is *algebraic,* i. e. belongs to an algebraic curve $f(x, y) = 0$, f-irreducible, then the valuation B is *algebraic* and is effectively of rank 2, being composed of the prime divisor defined by the prime ideal (f) and of a valuation of the field of rational functions on the curve $f(x, y) = 0$. In all cases, if B is algebroid, the induced valuation B^* of the field of meromorphic functions is of rank 2. If, namely, we denote by $P_0 = P, P_1, \cdots, P_{n-1}$

the n determinations of the power series $P(x)$, corresponding to the n determinations of $x^{1/n}$, then $\zeta = \prod_{i=0}^{n-1} (y - P_i)$ is an holomorphic function of x, y, i. e. an element of $\mathfrak{O}^*$ and is indecomposable in $\mathfrak{O}^*$. Given any element ξ of $\mathfrak{O}^*$, we have a unique decomposition $\xi = \zeta^\rho \xi_1$, $\xi_1 \not\equiv 0(\zeta)$. The substitution $y = P(x)$ does not annihilate ξ_1 and hence we find for ξ_1 a definite representation $\xi_1 = \gamma_1 x^{c_1} + \gamma_2 x^{c_2} + \cdots$, $\gamma_1 \neq 0$. We define B^* by putting $v(\xi) = (\rho, c_1)$.

It is evident that, conversely, any valuation B of Σ of rank 2 is algebraic, and if the values of x and y are positive, B can be defined by putting y equal to a Puiseux series in x, *provided that the divisor of which B is composed is of the first kind with respect to* $\mathfrak{O}$.

In the sequel we will have no occasion to use transcendental valuations. The results of Part I enable us, in fact, to prove the following theorem:

THEOREM 8. 1. *Every valuation ideal in $\mathfrak{O}$ belongs to an algebroid (and even to an algebraic) valuation of Σ.*

Proof.[9] It is sufficient to prove this assertion for 0-dimensional (primary) v-ideals, because v-ideals possessing a 1-dimensional component can belong only to algebraic valuations. Let $\{\mathfrak{q}_i\}$ be the sequence of zero-dimensional v-ideals in $\mathfrak{O}$ belonging to a valuation B. We wish then to prove that given any ideal in the sequence, say $\mathfrak{q}_n$, there exists an algebraic valuation for which $\mathfrak{q}_n$ is a valuation ideal. The nature of our proof requires that a stronger assertion be established. We propose to prove that *there exists an algebraic valuation $\bar{B}$ such that in the Jordan sequence $\{\bar{\mathfrak{q}}_i\}$ of the 0-dimensional v-ideals belonging to $\bar{B}$, the first n ideals $\bar{\mathfrak{q}}_1, \bar{\mathfrak{q}}_2, \cdots, \bar{\mathfrak{q}}_n$ coincide with $\mathfrak{q}_1, \mathfrak{q}_2, \cdots, \mathfrak{q}_n$.* This we prove by induction with respect to n, assuming then that this assertion has been already established for $n - 1$, *for any choice of the generators x, y of Σ and for any valuation of Σ whose valuation ring contains $\mathfrak{k}[x, y]$.* Assuming, as usual, that x and y have positive values in B, we use the quadratic transformation $T: x' = x$, $y' = y/x$, getting the ring $\mathfrak{O} = \mathfrak{k}[x', y']$ and the sequence of v-ideals $\{\mathfrak{q}'_i\}$ in $\mathfrak{O}'$ belonging to B. By our induction, there exists an algebraic valuation $\bar{B}$ of Σ whose valuation ring contains $\mathfrak{O}'$ and such that the ideals $\mathfrak{q}'_1, \mathfrak{q}'_2, \cdots, \mathfrak{q}'_{n-1}$ are v-ideals belonging to $\bar{B}$. Let $\{\bar{\mathfrak{q}}_i\}$ be the sequence of v-ideals in $\mathfrak{O}$ belonging to $\bar{B}$. We now use the results of section 4. The $n - 1$ ideals $T^{-1}\mathfrak{q}'_i$, $i = 1, 2, \cdots, n - 1$ must be members of both sequences $\{\mathfrak{q}_i\}$ and $\{\bar{\mathfrak{q}}_i\}$; here $T^{-1}\mathfrak{q}'_i$ is the contracted ideal of $x'^{h_i}\mathfrak{q}'_i$, where h_i is defined as the smallest integer such that $x'^{h_i}\mathfrak{q}'_i$ is an extended ideal of an ideal in $\mathfrak{O}$. Let $T^{-1}\mathfrak{q}'_i = \mathfrak{q}_{\alpha_i} = \bar{\mathfrak{q}}_{\alpha_i}$, $i = 1, 2, \cdots, n - 1$ (the indices α_i are the same, since

[9] Note that the proof makes no use of the Theorem 5. 2.

the index j of $\mathfrak{q}_j$ is its length). We also know that $\alpha_1 < \alpha_2 \cdots < \alpha_{n-1}$ and that $\alpha_{n-1} \geqq n$. Moreover, if $i \leqq \alpha_{n-1}$, then $\mathfrak{q}_i$ is necessarily of the form $\mathfrak{p}^\rho \mathfrak{q}_{\alpha_\sigma}$, $\sigma \leqq n-1$, and also $\bar{\mathfrak{q}}_i$ is of the form $\mathfrak{p}^{\bar{\rho}} \mathfrak{q}_{\alpha_{\bar{\sigma}}}$, $\bar{\sigma} \leqq n-1$. *We assert that* $\mathfrak{q}_i = \bar{\mathfrak{q}}_i$, $i = 1, 2, \cdots, \alpha_{n-1}$. Suppose that we know already that $\mathfrak{q}_j = \bar{\mathfrak{q}}_j$, for all $j < i \leqq \alpha_{n-1}$. If $\mathfrak{q}_i$ (or $\bar{\mathfrak{q}}_i$) coincides with one of the ideals $\mathfrak{q}_{\alpha_j}$, $j \leqq \alpha_{n-1}$, there is nothing to prove: we will have $\mathfrak{q}_i = \mathfrak{q}_{\alpha_\sigma} = \bar{\mathfrak{q}}_{\alpha_\sigma} = \bar{\mathfrak{q}}_i$. In the contrary case, we have $\mathfrak{q}_i = \mathfrak{p}^\rho \mathfrak{q}_{\alpha_\sigma}$, $\rho > 0$. It is not difficult to see that $\mathfrak{q}_i$ is then also of the form: $\mathfrak{q}_i = \mathfrak{p}\mathfrak{q}_j$, where necessarily $j < i$. In fact, let $\mathfrak{p}^{\rho-1}\mathfrak{q}_{\alpha_\sigma} \sim \mathfrak{q}_j$ (the equivalence being intended in the sense of the valuation B and $\mathfrak{q}_j$ being a v-ideal for B). Then $\mathfrak{p}^{\rho-1}\mathfrak{q}_{\alpha_\sigma} \equiv 0(\mathfrak{q}_j)$, whence $\mathfrak{q}_i \equiv 0(\mathfrak{p}\mathfrak{q}_j)$. On the other hand it is clear that $\mathfrak{q}_i \sim \mathfrak{p}\mathfrak{q}_j$, whence $\mathfrak{p}\mathfrak{q}_j \equiv 0(\mathfrak{q}_i)$. Hence $\mathfrak{q}_i = \mathfrak{p}\mathfrak{q}_j$. In a similar manner we find for $\bar{\mathfrak{q}}_i$ a representation of the form $\bar{\mathfrak{q}}_i = \mathfrak{p}\bar{\mathfrak{q}}_\mu$. Since $\mu < i$, we have $\bar{\mathfrak{q}}_\mu = \mathfrak{q}_\mu$, whence of the two ideals $\mathfrak{q}_j$ and $\bar{\mathfrak{q}}_\mu$ one is a divisor of the other ($\mathfrak{q}_j \equiv 0(\bar{\mathfrak{q}}_\mu)$ or $\bar{\mathfrak{q}}_\mu \equiv 0(\mathfrak{q}_j)$ according as $j \geqq \mu$ or $\mu \geqq j$). As a consequence it is also true that of the two ideals $\mathfrak{q}_i$ and $\bar{\mathfrak{q}}_i$, one is a divisor of the other. Now both $\mathfrak{q}_i$ and $\bar{\mathfrak{q}}_i$ have the same length i. Consequently $\mathfrak{q}_i = \bar{\mathfrak{q}}_i$, and this proves our assertion. Since the equality $\mathfrak{q}_i = \bar{\mathfrak{q}}_i$ holds for $i = 1, 2, \cdots, \alpha_{n-1}$ and since $\alpha_{n-1} \geqq n$, it follows that $\mathfrak{q}_1, \mathfrak{q}_2, \cdots, \mathfrak{q}_n$ belong as v-ideals to the algebraic valuation $\bar{B}$, and this proves our theorem.

Using Theorem 8.1 and the results of sections 5, 6, it is possible to give a very simple proof of the well-known fact that *every valuation B of Σ is the limit of algebraic valuations.* Let $\{\mathfrak{q}_i\}$ be the sequence of 0-dimensional v-ideals belonging to B. We may assume that the intersection of the ideals $\mathfrak{q}_i$ is the 0-ideal, since otherwise B itself is algebraic. In the proof of the Theorem 8.1 it has been shown that for any value of k there exists an algebraic valuation B_k of Σ, for which the ideals $\mathfrak{q}_1, \mathfrak{q}_2, \cdots, \mathfrak{q}_k$ are valuation ideals. Let r/s be any element of the valuation ring $\mathfrak{B}$ of B, $r, s \,\varepsilon\, \mathfrak{O}$. There will then exist an integer n such that $r \equiv 0(\mathfrak{q}_n)$, $s \not\equiv 0(\mathfrak{q}_{n+1})$. This integer will depend only on s. For all values of k such that $k \geqq n+1$, the ideals $\mathfrak{q}_n$ and $\mathfrak{q}_{n+1}$ will also be v-ideals for B_k, and hence for all such values of k r/s will also belong to the valuation ring $\mathfrak{B}_k$ of B_k. As a consequence, we have $\lim_{k\to\infty} (\mathfrak{B}_k \cap \mathfrak{B}) = \mathfrak{B}$, i. e. B can be regarded as the limit of the valuation B_k.

Using the characteristic property of the Jordan sequence $\{\mathfrak{q}_i\}$ of v-ideals established in section 2, and the properties of simple ideals given in sections 5 and 6, we may go a step further and gain an insight into the manner in which transcendental valuations are constructed. Given a simple v-ideal $\mathfrak{P}_{k+1}$ we know that it determines uniquely the sequence of simple v-ideals $\mathfrak{P}_1, \mathfrak{P}_2, \cdots, \mathfrak{P}_k$ which precede $\mathfrak{P}_{k+1}$ in any valuation to which $\mathfrak{P}_{k+1}$ belongs. We ask now the following question: given $\mathfrak{P}_k$ (and hence given the entire sequence $\mathfrak{P}_1, \mathfrak{P}_2, \cdots, \mathfrak{P}_k$)

in how many ways is it possible to choose $\mathfrak{P}_{k+1}$? To answer this question, we apply $k-1$ successive quadratic transformations, getting a ring $\bar{\mathfrak{O}}$ of polynomials of variables X, Y, in which to $\mathfrak{P}_k$ there corresponds a simple v-ideal $\bar{\mathfrak{P}}_1$ of kind one, i. e. $\bar{\mathfrak{P}}_1$ is prime and 0-dimensional, say $\bar{\mathfrak{P}}_1 = (X, Y)$. Any maximal subideal of $\bar{\mathfrak{P}}_1$ is of the form $(aX + bY, \bar{\mathfrak{P}}_1{}^2)$, where a, b are in $\mathfrak{k}$ and are not both zero. It is obvious that any such maximal subideal of $\bar{\mathfrak{P}}_1$ belongs to some valuation (for instance, to any valuation defined by putting $X = bt + \cdots, Y = -at + \cdots$) and is moreover a simple v-ideal. These maximal subideals are in $(1, 1)$ correspondence with the ratio $z = a/b$, i. e. with the places of the purely transcendental field $\mathfrak{k}(z)$. Going back to our original ring $\mathfrak{O}$, we see that the set of simple v-ideals $\mathfrak{P}_{k+1}$ of kind $k+1$ such that $\mathfrak{P}_1, \mathfrak{P}_2, \cdots, \mathfrak{P}_k, \mathfrak{P}_{k+1}$ belong to one and the same valuation, is in $(1, 1)$ correspondence with the set consisting of the elements of the underlying field $\mathfrak{k}$ and of the symbol ∞. Starting with $\mathfrak{P}_1$ we can then construct, in infinitely many ways, an infinite sequence $\mathfrak{P}_1, \mathfrak{P}_2, \cdots, \mathfrak{P}_k, \cdots$ of simple v-ideals, such that, for any k, the ideals $\mathfrak{P}_1, \mathfrak{P}_2, \cdots, \mathfrak{P}_k$ belong to some valuation B_k, which we may suppose to be algebraic. We assert that *the infinite sequence* $\{\mathfrak{P}_k\}$ *defines a valuation* B *of* Σ. Obviously then $B = \operatorname{Lim} B_k$. To see this, we first observe, that by Theorem 6. 2, the infinite sequence $\{\mathfrak{P}_k\}$ determines uniquely an infinite Jordan sequence $\{\mathfrak{q}_j\}$ which contains the sequence $\{\mathfrak{P}_k\}$ and which has the property that the elements of the sequence which precede a given $\mathfrak{P}_i$ are v-ideals for all the valuations B_k, $k \geqq i$. It follows immediately that the congruence $\mathfrak{q}_{m+1}\mathfrak{q}_n : \mathfrak{q}_m \equiv 0(\mathfrak{q}_{n+1})$ holds true for any two ideals $\mathfrak{q}_m, \mathfrak{q}_n$ of the sequence $\{\mathfrak{q}_j\}$, since the ideals which occur in this congruence are v-ideals belonging to B_k, when k is sufficiently large. As a consequence the sequence effectively defines a valuation of Σ, *q. e. d.*

9. Valuation ideals in the ring of holomorphic functions.[10] It has been pointed out in the preceding section that any 0-dimensional valuation B of the field $\Sigma = \mathfrak{k}(x, y)$ in which x and y have positive values, defines a valuation B^* of the field Σ^* of meromorphic functions of x, y, whose valuation ring contains the ring $\mathfrak{O}^* = \mathfrak{k}\{x, y\}$ of holomorphic functions of x, y. We have then also valuation ideals in $\mathfrak{O}^*$ belonging to B^*. It is clear that the prime ideal defined by B^* in $\mathfrak{O}^*$ is the 0-dimensional ideal $\mathfrak{p}^* = (x, y)$, i. e. the extended ideal of the ideal $\mathfrak{p} = (x, y)$ in $\mathfrak{O}$. Let $\{\mathfrak{q}_i\}$ be the sequence of 0-dimensional v-ideals in $\mathfrak{O}$ belonging to B, and let $\mathfrak{q}^*_i = \mathfrak{O}^*\mathfrak{q}_i$ be the extended

[10] Results of this and of the following sections will be later applied toward the proof of Theorem 5. 2. The properties of simple v-ideals derived in Part I and based on Theorem 5. 2 are therefore not to be used until a proof of this theorem has been given (Corollary 11. 2). We may, however, use Theorem 8. 1 (see footnote on p. 175).

ideal of $\mathfrak{q}_i$ in $\mathfrak{O}^*$. It is clear that $v(q^*_{i+1}) > v(\mathfrak{q}^*_i)$ in B^*, since $v(\mathfrak{q}^*_i) = v(\mathfrak{q}_i)$. Let $f_1, f_2, \cdots, f_k$ be a base of $\mathfrak{q}_i$ and let f be an element of $\mathfrak{q}_i$ not in $\mathfrak{q}_{i+1}$. Since $\mathfrak{q}_i/\mathfrak{q}_{i+1} \cong \mathfrak{k}$, we have $f_i \equiv c_i f(\mathfrak{q}_{i+1})$, $c_i \varepsilon \mathfrak{k}$. If then $\xi = \xi_1 f_1 + \cdots + \xi_k f_k$ is any element of $\mathfrak{q}^*_i$ ($\xi_j \varepsilon \mathfrak{O}^*$), we have $\xi \equiv f \cdot (c_1\xi_1 + \cdots + c_k\xi_k)(\mathfrak{q}^*_{i+1})$. Now $c_1\xi_1 + \cdots + c_k\xi_k \equiv c(\mathfrak{p}^*)$, where $c \varepsilon \mathfrak{k}$, and

$$(f)\mathfrak{p}^* \equiv 0(\mathfrak{q}^*_i\mathfrak{p}^*) \equiv 0(\mathfrak{O}^*\mathfrak{q}_i\mathfrak{p}) \equiv 0(\mathfrak{q}^*_{i+1}),$$

since $\mathfrak{q}_i\mathfrak{p} \equiv 0(\mathfrak{q}_{i+1})$. Hence $\xi \equiv cf(\mathfrak{q}^*_{i+1})$, and this shows that $\mathfrak{q}^*_i/\mathfrak{q}^*_{i+1} \simeq \mathfrak{k}$, i. e. $\mathfrak{q}^*_{i+1}$ is a maximal subideal of $\mathfrak{q}^*_i$. Hence the sequence $\{\mathfrak{q}^*_i\}$ is a Jordan sequence in $\mathfrak{O}^*$, and since $v(\mathfrak{q}^*_{i+1}) > v(\mathfrak{q}^*_i)$, the sequence $\{\mathfrak{q}^*_i\}$ is the sequence of v-ideals in $\mathfrak{O}^*$ belonging to B^*, i. e. *the zero-dimensional v-ideals in $\mathfrak{O}^*$ belonging to B^* are the extended ideals of the zero-dimensional v-ideals in $\mathfrak{O}$ belonging to B.* It is of course evident that $\mathfrak{q}_i$ is the contracted ideal of $\mathfrak{q}^*_i$.

The theorems of sections 3, 4,[11] the notions of simple and composite ideals introduced in section 5 and Theorem 5.1 carry over without modification to ideals in $\mathfrak{O}^*$. In order to see this, a perusal of the proof is not at all necessary. It is sufficient to take into account quite generally the nature of the relationship between the polynomial ideals in $\mathfrak{O}$ and the power series ideals in $\mathfrak{O}^*$. There is a $(1, 1)$ correspondence between primary 0-dimensional ideal in $\mathfrak{O}$ belonging to the prime ideal $\mathfrak{p} = (x, y)$ and the 0-dimensional ideals in $\mathfrak{O}^*$. The correspondence is such that to an ideal $\mathfrak{q}$ in $\mathfrak{O}$ there corresponds its extended ideal $\mathfrak{q}^*$ in $\mathfrak{O}^*$ and $\mathfrak{q}$ is the contracted ideal of $\mathfrak{q}^*$. In fact, let $\mathfrak{q}^*$ be any 0-dimensional ideal in $\mathfrak{O}^*$. If ρ is the exponent of $\mathfrak{q}^*$, the $\mathfrak{q}^*$ possesses a base consisting of the power products $x^i y^j$, $i + j = \rho$, and of a set of polynomials F_a of degree $< \rho$. Hence $\mathfrak{q}^* = (F_a, x^i y^j)$ and therefore $\mathfrak{q}^*$ is the extended ideal of a zero-dimensional primary polynomial ideal $\mathfrak{q}$ belonging to $\mathfrak{p}$ $(= (x, y))$. Let $\mathfrak{q}^* = (f_1, f_2, \cdots, f_k)$, where $f_1, f_2, \cdots, f_k$ is a base of $\mathfrak{q}$, and let $\mathfrak{q}'$ be the contracted ideal in $\mathfrak{O}$ of $\mathfrak{q}^*$. If F is any polynomial in $\mathfrak{q}'$, $F = \sum_{i=1}^{k} \xi_i(x, y) f_i$, $\xi_i \varepsilon \mathfrak{O}^*$, and if we denote by $A_i^{(n)}$ the partial sum of the terms of degree $\leqq n$ in the power series $\xi_i(x, y)$, then the polynomial $F - \Sigma A_i^{(n)} f_i$ does contain terms of degree $< n + 1$, whence $F \equiv \Sigma A_i^{(n)} f_i (\mathfrak{p}^{n+1})$. Now, if n is sufficiently high, then $\mathfrak{p}^{n+1} \equiv 0(\mathfrak{q})$, whence $F \equiv 0(\mathfrak{q})$. Hence $\mathfrak{q}' \equiv 0(\mathfrak{q})$, and since $\mathfrak{q}'$ is the contracted ideal of $\mathfrak{q}^*$, it follows $\mathfrak{q}' = \mathfrak{q}$. Thus every zero-dimensional ideal $\mathfrak{q}^*$ in $\mathfrak{O}^*$ is the extended ideal of one and only

[11] In the case of formal power series the quadratic transformation $x' = x$, $y' = y/x$ leads from the ring $k\{x, y\}$ of formal power series in x and y to the larger ring of $k\{x', y'\}$ of formal power series in x' and y'.

one primary ideal $\mathfrak{q}$ in $\mathfrak{O}^*$, and $\mathfrak{q}$ is the contracted ideal of $\mathfrak{q}^*$. In particular, $\mathfrak{q}^*$ is simple or composite, according as $\mathfrak{q}$ is simple or composite.

10. The notion of a general element of an ideal of formal power series. The ring $\mathfrak{O}^*$ of holomorphic functions of x and y contains only one prime 0-dimensional prime ideal, namely the ideal $\mathfrak{p}^* = (x, y)$, and every 0-dimensional ideal $\mathfrak{A}$ in $\mathfrak{O}^*$ is necessarily primary and belongs to $\mathfrak{p}^*$. Since any ideal in $\mathfrak{O}^*$ has a finite base, given a 0-dimensional ideal $\mathfrak{A}$, it belongs to a finite exponent ρ, i. e. $\mathfrak{p}^{*\rho} \equiv 0(\mathfrak{A})$. In other words, $\mathfrak{A}$ contains all the formal power series which contain terms of lowest degree $\geqq \rho$. Since $\mathfrak{A}$ is at any rate a linear $\mathfrak{k}$-module, it follows that the condition in order that an element $\xi = \sum_{i,j \geqq 0} a_{ij}, x^i y^j$ of $\mathfrak{O}^*$ belong to $\mathfrak{A}$ is expressed by linear homogeneous relations between the coefficients a_{ij}, $i + j \leqq \rho$. With every linear relation

$$\sum_{i,j<\rho} c_{ij} a_{ij} = 0, \qquad c_{ij} \varepsilon \mathfrak{k}, \tag{17}$$

satisfied by all the elements ξ of $\mathfrak{A}$, we associate the function

$$E = \sum c_{ij} x^{-i} y^{-j}. \tag{17'}$$

The set of all the functions E obtained in this manner is called the "inverse system" $\mathfrak{A}^{-1}$ of the ideal $\mathfrak{A}$ (Macaulay,[5,6] Lasker[3]). If $E_1, E_2, \cdots, E_r$ belong to $\mathfrak{A}^{-1}$, then also $\alpha_1 E_1 + \cdots + \alpha_r E_r$ is in $\mathfrak{A}^{-1}$, $\alpha_1, \cdots, \alpha_r \varepsilon \mathfrak{k}$. From the fact that if $\xi = \Sigma a_{ij} x^i y^j$ belongs to the ideal $\mathfrak{A}$, also

$$x\xi = \Sigma a_{ij} x^{i+1} y^j \quad \text{and} \quad y\xi = \Sigma a_{ij} x^i y^{j+1}$$

belong to $\mathfrak{A}$, it follows immediately that if $E = c_{ij} x^{-i} y^{-j}$ is an element of the inverse system, then also the following relations are true for any element $\xi = \Sigma a_{ij} x^i y^j$ in $\mathfrak{A}$:

$$\sum c_{i+1,j} a_{ij} = 0, \qquad \sum c_{i,j+1} a_{ij} = 0.$$

This shows that the inverse system $\mathfrak{A}^{-1}$ *becomes an* $\mathfrak{O}^*$*-module, provided that we define multiplication as follows:*

$$x^\alpha y^\beta \sum_{i,j \geqq 0} c_{ij} x^{-i} y^{-j} = \sum_{\substack{i \geqq \alpha \\ j \geqq \beta}} c_{ij} x^{-i+\alpha} y^{-j+\beta}, \qquad \alpha, \beta \geqq 0.$$

In fact, with this definition of multiplication by power products $x^\alpha y^\beta$, the above

equations $\Sigma c_{i+1,j}a_{ij}=0$, $\Sigma c_{i,j+1}a_{ij}=0$ signify that if E is in $\mathfrak{A}^{-1}$, also xE and yE are in $\mathfrak{A}^{-1}$. Multiplication of E by any formal power series in $\mathfrak{O}^*$ is then to be defined formally by the requirement of the distributive law of multiplication. It is then not difficult to see that $\mathfrak{A}^{-1}$ *consists of those and only those functions E which have the property that $\xi E=0$ for any element ξ in $\mathfrak{A}$.* It is evident that if $\mathfrak{A}$ and $\mathfrak{B}$ are ideals in $\mathfrak{O}^*$, then $\mathfrak{A}^{-1}=\mathfrak{B}^{-1}$ implies $\mathfrak{A}=\mathfrak{B}$. Moreover, it is not difficult to show that given any set $\mathfrak{C}^{-1}$ of functions E which is an $\mathfrak{O}^*$-module and for which there exists at least one element $\xi\neq 0$ in $\mathfrak{O}^*$ such that $\xi E=0$ for any E in $\mathfrak{C}^{-1}$, then the set of all elements ξ satisfying this condition forms an ideal $\mathfrak{A}$, and $\mathfrak{C}^{-1}$ is the inverse system of $\mathfrak{A}$.

We now come to the definition of a concept which shall be very useful in the sequel. Let $\tau=t_{ij}x^iy^j$ be a formal power series whose coefficients t_{ij} belong to an extension field $\mathfrak{K}$ of $\mathfrak{k}$. We shall say that τ is a *variable element of an ideal* $\mathfrak{A}$, if the coefficients t_{ij} do not all belong already to $\mathfrak{k}$ (so that at least one of the t_{ij}'s is transcendental with respect to $\mathfrak{k}$) and if they satisfy (in $\mathfrak{K}$) all the linear relations (17) (with a_{ij} replaced by t_{ij}), which are satisfied by the coefficients of all the actual power series belonging to $\mathfrak{A}$. In other words, we require that $\tau E=0$ for any function E in $\mathfrak{A}^{-1}$. We shall say that a variable element τ of $\mathfrak{A}$ is a *general element of* $\mathfrak{A}$, if the t_{ij}'s do not satisfy any other algebraic relation, algebraically independent of the above linear relations. Finally, we shall say that a variable element τ of $\mathfrak{A}$ is a *quasi-general* element of $\mathfrak{A}$, if the t_{ij}'s do not satisfy *linear relations* other than those which are satisfied by the coefficients of a general element of $\mathfrak{A}$.

In a similar manner we define the notions of a variable element, of a general element and of a quasi-general element of the inverse system $\mathfrak{A}^{-1}$. The condition that a function $E=\Sigma c_{ij}x^{-i}y^{-j}$ belong to $\mathfrak{A}^{-1}$ is expressed by a certain set (δ) of homogeneous linear equations between the c_{ij}'s. These equations are obtained by expressing the fact that $E\xi_i=0$, $i=1,2,\cdots,k$, where $\xi_1,\xi_2,\cdots,\xi_k$ is a base of the ideal $\mathfrak{A}$. A function $E=\Sigma e_{ij}x^{-i}y^{-j}$, where the e_{ij} are elements of an extension field $\mathfrak{k}(e_{ij})$ of $\mathfrak{k}$, shall be called a *variable element of* $\mathfrak{A}^{-1}$, if the coefficients e_{ij} satisfy all the relations of the set (δ), i. e. if $E\xi=0$ for any ξ in $\mathfrak{A}$. If the coefficients e_{ij} do not satisfy algebraic (*or* linear) relations algebraically independent of the linear relations (δ), then E shall be called a *general* (*or quasi-general*) element of $\mathfrak{A}^{-1}$.

Let $E=\Sigma e_{ij}x^{-i}y^{-j}$ and $\tau=\Sigma t_{ij}x^iy^j$ be variable elements of $\mathfrak{A}^{-1}$ and of $\mathfrak{A}$ respectively, where we assume that the e_{ij} and t_{ij} are elements of one and the same field $\mathfrak{K}$.

We have $\xi E=0$ for any element $\xi=\Sigma a_{ij}x^iy^j$ of $\mathfrak{A}$. Since the a_{ij}'s are arbitrary elements of underlying field $\mathfrak{k}$ subject to the only condition of satis-

fying a finite set of linear relations $\Sigma c_{ij}a_{ij}=0$, where $\Sigma c_{ij}x^{-i}y^{-j}$ is an element of $\mathfrak{A}^{-1}$, it follows that the relation $\xi E=0$ remains true if we regard the a_{ij} as indeterminates connected by the linear relations $\Sigma c_{ij}a_{ij}=0$. But then the relation $\xi E=0$ holds also after the specialization $a_{ij}\rightarrow t_{ij}$, since the t_{ij} also satisfy all the relations $\Sigma c_{ij}t_{ij}=0$. *We conclude that* $\tau E=0$. The following assertion is now easily derived:

If E is a quasi-general element of $\mathfrak{A}^{-1}$, then $\tau E=0$ implies that τ is a variable element of $\mathfrak{A}$, and if τ is a quasi-general element of $\mathfrak{A}$ then $\tau E=0$ implies that E is a variable element of $\mathfrak{A}^{-1}$. Proof straightforward.

Let $\mathfrak{A}', \mathfrak{A}''$ be two (distinct or coincident) ideals in $\mathfrak{O}^*$ and let $\tau'=\Sigma t'_{ij}x^iy^j$, $\tau''=\Sigma t''_{ij}x^iy^j$ be variable elements of $\mathfrak{A}'$ and $\mathfrak{A}''$ respectively. Let $\mathfrak{K}'=\mathfrak{k}(t'_{ij})$ and $\mathfrak{K}''=\mathfrak{k}(t''_{ij})$. We consider a pure transcendental base $\{u\}$ of $\mathfrak{K}''$ over $\mathfrak{k}$ and we adjoin the elements of this base to the field $\mathfrak{K}'$, *adjunction to be regarded as a pure transcendental extension of* $\mathfrak{K}'$. We obtain in this manner the field $\mathfrak{K}'(\{u\})$ having $\mathfrak{k}(\{u\})$ as subfield, and we then adjoin to $\mathfrak{K}'(\{u\})$ all the elements of $\mathfrak{K}''$ which are algebraic with respect to $\mathfrak{k}(\{u\})$. In the field $\mathfrak{K}=\mathfrak{k}(t'_{ij}, t''_{ij})$ obtained in this manner, any algebraic relation between the t'_{ij} and the t''_{ij} must be a consequence of algebraic relations between the t'_{ij}'s alone in $\mathfrak{K}'$ and the t''_{ij} alone in $\mathfrak{K}''$. Now that the t'_{ij} and the t''_{ij} have been properly imbedded in a common field, we form the product $\tau'\tau''=\tau=\Sigma t_{ij}x^iy^j$, $t_{ij}\in\mathfrak{K}$, and we refer to τ as *the direct product of the variable elements* τ', τ'' *of* $\mathfrak{A}'$ *and* $\mathfrak{A}''$.

THEOREM 10. 1. *The direct product $\tau=\tau'\tau''$ of variable elements of two ideals $\mathfrak{A}', \mathfrak{A}''$ is a variable element of the product $\mathfrak{A}'\mathfrak{A}''$ of the two ideals. If τ', τ'' are general or quasi-general elements of $\mathfrak{A}'$ and $\mathfrak{A}''$ respectively, then τ is a quasi-general* (*not necessarily general*) *element of* $\mathfrak{A}'\mathfrak{A}''$.

Proof. Let $\mathfrak{A}'\mathfrak{A}''=\mathfrak{B}$. It is well known (and is of immediate verification) that $\mathfrak{B}^{-1}=\mathfrak{A}'^{-1}:\mathfrak{A}''$, i. e. $\mathfrak{B}^{-1}$ consists of all functions E such that $E\mathfrak{A}''\in\mathfrak{A}'^{-1}$. Let then $E_0=\Sigma c_{ij}x^{-i}y^{-j}$ be an element in $\mathfrak{B}^{-1}$ and let $\tau''E_0=\Sigma\sigma_{ij}x^{-i}y^{-j}$, $\sigma_{ij}\in\mathfrak{K}$. *We assert that $\tau''E_0$ is a variable element of $\mathfrak{A}'^{-1}$.* In fact, consider any element $\xi''=\Sigma a''_{ij}x^iy^j$ of $\mathfrak{A}''$. Since $E_0\mathfrak{A}''\in\mathfrak{A}'^{-1}$, the product $\xi''E_0=\Sigma\sigma_{ij}^{(0)}x^{-i}y^{-j}$ is an element of the inverse system $\mathfrak{A}'^{-1}$. Hence the coefficients $\sigma_{ij}^{(0)}$ must satisfy the linear relations (δ'') which are satisfied by the coefficients of the general elements of $\mathfrak{A}'^{-1}$. The coefficients $\sigma_{ij}^{(0)}$ being linear forms in the coefficients a''_{ij} of ξ'', we get by substitution into the equations (δ'') a set of linear homogeneous relations between the a''_{ij}. Since these relations hold true for the coefficients a''_{ij} of any element ξ'' of $\mathfrak{A}''$,

it follows that they are also satisfied by the coefficients t''_{ij} of the variable element τ'' of $\mathfrak{A}''$. But then it follows that the coefficients σ_{ij} of $\tau''E_0$ must satisfy the linear equations (δ''), whence $\tau''E_0$ is a variable element of $\mathfrak{A}'^{-1}$. Since τ' is a variable element of $\mathfrak{A}'$, it follows by a previously proved result, that $\tau'\tau''E_0 = 0$, i. e. $\tau E_0 = 0$. Since the relation $\tau E_0 = 0$ holds for any element E_0 of $\mathfrak{B}^{-1}$, it follows that τ is a variable element of $\mathfrak{B}$, and this proves the first part of the theorem.

Now suppose that τ' and τ'' are general (or quasi-general) elements of the ideals $\mathfrak{A}'$ and $\mathfrak{A}''$ respectively. To prove that in this case τ is a quasi-general element of $\mathfrak{B}$ we have to show that if $\Sigma c_{ij}t_{ij} = 0$ is any linear homogeneous relation between the t_{ij}, then $E = \Sigma c_{ij}x^{-i}y^{-j}$ belongs to the inverse system $\mathfrak{B}^{-1}$. Now the t_{ij} are bilinear forms in the two sets of coefficients t'_{ij} and t''_{ij}. Substituting these bilinear forms we get $\Sigma c_{ij}t_{ij} = H(t'_{ij}, t''_{ij})$, where H is a bilinear form in the t'_{ij} and the t''_{ij}. The relation $H(t'_{ij}, t''_{ij}) = 0$ must be a consequence of the *linear relations* between the t'_{ij} and the t''_{ij} separately, since τ is the direct product of τ' and τ''. Now τ' is a quasi-general element of $\mathfrak{A}'$. As a consequence, any *linear* relation between the t'_{ij} arises from a function E' in $\mathfrak{A}'^{-1}$ and is therefore not destroyed if each t'_{ij} is replaced by $t'_{i-1,j}$ or by $t'_{i,j-1}$ (multiplication of E' by x or by y respectively). Hence the relation $H(t'_{ij}, t''_{ij}) = 0$ implies the relations $H(t'_{i-1,j}, t''_{ij}) = 0$ and $H(t'_{i,j-1}, t''_{ij}) = 0$. It is clear that these two relations correspond to the relations $\Sigma c_{ij}t_{i-1,j} = 0$ and $\Sigma c_{ij}t_{i,j-1} = 0$, which therefore must be true relations between the t_{ij}. As a consequence the functions $E = c_{ij}x^{-i}y^{-j}$ corresponding to the various linear relations $\Sigma c_{ij}t_{ij} = 0$ between the t_{ij}, form an $\mathfrak{O}^*$-module. From this it follows immediately that $E\tau = 0$, for any E in this module. Now $E\tau = E\tau'\tau'' = 0$ implies that $E\tau'$ is a variable element of $\mathfrak{A}''^{-1}$, since τ'' is a quasi-general element of $\mathfrak{A}''$. The condition of belonging to $\mathfrak{A}''^{-1}$ is expressed by a certain set (δ'') of linear equations, and thus the coefficients of $E\tau'$ must satisfy these equations. Since τ' is a quasi-general element of $\mathfrak{A}'$, these linear equations (δ'') must be satisfied by $E\xi'$, ξ'—an arbitrary element of $\mathfrak{A}'$, and hence $E\mathfrak{A}' \varepsilon \mathfrak{A}''^{-1}$. As a consequence $E \varepsilon \mathfrak{A}''^{-1} : \mathfrak{A}' = \mathfrak{B}^{-1}$, which proves our theorem.[12]

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[12] That τ need not be the general element of $\mathfrak{A}'\mathfrak{A}''$ is shown by the following example. Let $\mathfrak{p} = (x, y)$, $\mathfrak{q} = (x^2, y^2)$, so that $\mathfrak{p}\mathfrak{q} = \mathfrak{p}^3$. The general element τ' of $\mathfrak{p}$ is $t'_{10}x + t'_{01}y + t'_{20}x^2 + \cdots$, where the t'_{ij}'s are indeterminates. Similarly

$$\tau'' = t''_{20}x^2 + t''_{02}y^2 + t''_{30}x^3 + \cdots$$

is the general element of $\mathfrak{q}$. Now

$$\tau = \tau'\tau'' = t_{30}x^3 + t_{21}x^2 + t_{12}xy^2 + t_{03}y^3 + \cdots,$$

where the t_{ij} satisfy *one non-linear* relation $t_{30}t_{03} = t_{21}t_{12}$.

The preceding theorem implies that if an ideal $\mathfrak{B}$ is composite, then a suitable quasi-general element τ of $\mathfrak{B}$ (namely the direct product of the general elements of the factors of $\mathfrak{B}$) is reducible in the algebraic closure of the field of the coefficients t_{ij} of τ. We are interested in the question of the extent to which the above considerations can be inverted. *What can be said about an ideal $\mathfrak{B}$, if its general element τ is reducible in the algebraic closure of the coefficients of τ?* That the ideal $\mathfrak{B}$ need not in general be composite, is illustrated by the example $\mathfrak{B} = (x^2, y^2)$. The ideal $\mathfrak{B}$ is simple, but its general element

$$\tau = t_{20}x^2 + t_{02}y^2 + t_{30}x^3 + \cdots \; (t_{00} = t_{10} = t_{01} = t_{11} = 0)$$

is reducible, since

$$\tau = (\sqrt{t_{20}}x + \sqrt{-t_{02}}y + \cdots)(\sqrt{t_{20}}x - \sqrt{-t_{02}}y + \cdots).$$

However, in the special case of valuation ideals we can prove the following theorem:

THEOREM 10.2. *If $\mathfrak{A}$ is a valuation ideal and if the general element t of $\mathfrak{A}$ is reducible, $t = t_1 t_2 \cdots t_k$ (in the algebraic closure of its coefficients), then $\mathfrak{A}$ is composite; t is the direct product of its irreducible factors t_i, each irreducible factor t_i is the general element of a valuation ideal $\mathfrak{A}_i$, and $\mathfrak{A} = \mathfrak{A}_1\mathfrak{A}_2 \cdots \mathfrak{A}_k$.*

Proof. Let $t = \Sigma t_{ij}x^i y^j$ and let $t = t_1 t_2 \cdots t_k$ be the factorization of t into irreducible factors $t_i = t_i(x, y)$, belonging to the ring of formal power series of x, y with coefficients in the algebraic closure of $\mathfrak{k}(\cdots t_{ij} \cdots)$. We assume of course that no t_i is a unit, i. e. that t_i does not contain a term independent of x and y. The valuation B to which $\mathfrak{A}$ belongs can be assumed to be algebraic. Substituting into t and into t_i the Puiseux expansion $y = P(x)$ which determines the valuation B, we are able to attach to these elements definite values $v(t)$, $v(t_i)$. It is clear that $v(t) = v(\mathfrak{A})$. Let $v(t_i) = \alpha_i$, whence $v(t) = \alpha_1 + \alpha_2 + \cdots + \alpha_k$. There exist elements in $\mathfrak{k}\{x, y\}$ whose value in B equals α_i: they can be obtained by specializing the coefficients of the formal power series t_i in such a manner as not to annihilate the coefficient of the leading term x^{α_i} of $t_i(x, P(x))$. As a consequence there exists a v-ideal $\mathfrak{A}_i$ for B such that $v(\mathfrak{A}_i) = \alpha_i$. Since $\alpha_i > 0$, no $\mathfrak{A}_i$ is the unit ideal. Since $v(t_i) = v(\mathfrak{A}_i)$, it follows that t_i is a variable element of $\mathfrak{A}_i$, and hence $t = \Pi t_i$

is a variable element of the ideal $\mathfrak{A}_1\mathfrak{A}_2\cdots\mathfrak{A}_k$. But since t is a general element of $\mathfrak{A}$, necessarily $\mathfrak{A}\equiv 0(\mathfrak{A}_1\mathfrak{A}_2\cdots\mathfrak{A}_k)$. On the other hand we have

$$v(\mathfrak{A})=v(t)=\Sigma v(t_i)=\Sigma v(\mathfrak{A}_i)=v(\Pi\mathfrak{A}_i).$$

Hence $\Pi\mathfrak{A}_i\equiv 0(\mathfrak{A})$, and consequently $\mathfrak{A}=\mathfrak{A}_1\mathfrak{A}_2\cdots\mathfrak{A}_k$.

Let τ_i be the general element of $\mathfrak{A}_i$ and let us consider the *direct* product $\tau=\tau_1\tau_2\cdots\tau_k$. By Theorem 10. 1, τ is a variable element of $\mathfrak{A}$. On the other hand, any algebraic relation between the coefficients of the power series τ leads also to a true relation between the coefficients of t, since t is obtained from τ by the specialization $\tau_i\rightarrow t_i$. Consequently τ *must be a general element of* $\mathfrak{A}$. It can be then identified with t, and it is thus seen that in the original factorization $t=\Pi t_i$, each t_i is a general element of $\mathfrak{A}_i$ and that the product is a direct product (unicity of factorization of t into irreducible factors). The theorem is proved.

COROLLARY. *The general element of a simple v-ideal is absolutely irreducible* (i. e. irreducible in $\mathfrak{K}\{x,y\}$, $\mathfrak{K}$ being any extension field of $\mathfrak{k}$).

11. The characterization of simple *v*-ideals. Let

$$(18)\qquad y=y_1=\sum_{i=1}^{k}c_ix^{\alpha_i/\nu}+\sum_{j=0}^{\infty}t_jx^{(\alpha_{k+1}+j)/\nu},\qquad 0<\alpha_1<\alpha_2<\cdots$$

be a Puiseux series, *in which we assume that* $(\nu,\alpha_1,\alpha_2,\cdots,\alpha_g)=1$, $g\leqq k+1$, *and that the first k coefficients $c_1,\cdots,c_k$ are in the underlying field $\mathfrak{k}$, while the remaining coefficients are indeterminates.*

THEOREM 11. 1. *Given a formal power series $\xi(x,y)=\Sigma a_{ij}x^iy^j$ with indeterminate coefficients a_{ij}, there exists a set of linear forms $F_m(a_{ij})$, $G_n(a_{ij})$ which have the following properties:* (1) *the relations $F_m(a_{ij}{}^{(0)})=0\,(a_{ij}{}^{(0)}\,\varepsilon\,\mathfrak{k})$ give necessary conditions that the equation $\xi_0(x,y)=\Sigma a_{ij}{}^{(0)}x^iy^j=0$ admit a uniformization of type* (18), *the t_j's being replaced by special values $t_j{}^{(0)}$ in $\mathfrak{k}$;* (2) *the relations $F_m(a_{ij}{}^{(0)})=0$ and the inequalities $G_n(a_{ij}{}^{(0)})\neq 0$ give necessary and sufficient condition in order that $\xi_0(x,y)=0$ admit the above uniformization and that $\xi_0(x,y)$ be an irreducible element of $\mathfrak{k}\{x,y\}$. The set of all elements ξ in $\mathfrak{k}\{x,y\}$ whose coefficients a_{ij} satisfy the relations $F_m=0$, is a simple ideal.*

Proof. The theorem is true for $g=0$, in which case $\nu=1$. In fact, let

$\bar{y}=\sum_{i=1}^{k} c_i x^{\alpha_i}$ ($\bar{y}=0$, if $k=0$). If the equation $\xi(x,y)=0$ is uniformizable by the expansion $y=y_1$, then we must have $\xi(x,y_1)=0$, and this shows that $\xi(x,\bar{y})$ is divisible by $x^{\alpha_{k+1}}$. This condition is expressed by a certain set of linear homogeneous equations $F_m(a_{ij})=0$ between the coefficients of $\xi(x,y)$. Should moreover $\xi(x,y)$ be irreducible, then we must have

$$\xi(x,y)=(y-y_1)\eta(x,y),$$

where $\eta(x,y)$ is a unit. As a consequence, the coefficient a_{01} must be different from 0. Conversely, if $F_m(a_{ij})=0$ and $a_{01}\neq 0$, then $\xi(x,y)$ is divisible by $y-y_1$ and is irreducible in $\mathfrak{k}\{x,y\}$.

We assume that the theorem is true for $g-1$. Let $\nu=\nu_1 n$, $\alpha_1=\nu_1\alpha'_1$, $(n,\alpha'_1)=1$. We put

(19) $$x=\bar{x}^n, \qquad y=\bar{x}^{\alpha'_1}(c_1+\bar{y}).$$

If $\xi(x,y)=\Sigma a_{ij}x^i y^j$ can be uniformized by the expansion $y=y_1$, then ξ must be divisible by the product $\prod_{i=1}^{\nu}(y-y_1^{(i)})$, where $y_1=y_1^{(1)},\cdots,y_1^{(\nu)}$ are the conjugates of y_1. From this it follows immediately that ξ cannot contain terms $x^i y^j$ in which $\nu i+\alpha_1 j<\nu\alpha_1$, whence

(20) $$a_{ij}=0, \text{ for all } i,j, \text{ such that } \nu i+\alpha_1 j<\nu\alpha_1.$$

Substituting (19) we find, in view of (20),

(20′) $$\xi(x,y)=\bar{x}^{\nu\alpha'_1}\bar{\xi}(\bar{x},\bar{y})=\bar{x}^{\nu\alpha'_1}\Sigma\bar{a}_{ij}\bar{x}^i\bar{y}^j,$$

while the equation of the branch (18) becomes

(21) $$\bar{y}=\bar{y}_1=\sum_{i=2}^{k} c_i\bar{x}^{(\alpha_i-\alpha_1)/\nu_1}+\sum_{j=0}^{\infty} t_j\bar{x}^{(\alpha_{k+1}+j-\alpha_1)/\nu_1}.$$

The equation $\bar{\xi}(\bar{x},\bar{y})=0$ admits a uniformization by means of an expansion $\bar{y}=\bar{y}_1$ of type (21). Since $\alpha_2>\alpha_1$, there can be no constant term in $\bar{\xi}(\bar{x},\bar{y})$. This constant term arises from the terms $a_{ij}x^i y^j$ of $\xi(x,y)$ in which $\nu i+\alpha_1 j=\nu\alpha_1$, i. e. $ni+\alpha'_1 j=n\alpha'_1\nu_1$, and is therefore equal to

$$a_{\nu_1\alpha'_1,0}+a_{(\nu_1-1)\alpha'_1,n}c_1^{\,n}+\cdots+a_{0,n\nu_1}c_1^{\,n\nu_1}.$$

Consequently

(22) $$a_{\nu_1\alpha'_1,0}+a_{(\nu_1-1)\alpha'_1,n}c_1^{\,n}+\cdots+a_{0,n\nu_1}c_1^{\,n\nu_1}=0.$$

It is clear that $(\alpha_2-\alpha_1,\alpha_3-\alpha_1,\cdots,\alpha_g-\alpha_1,\nu_1)=1$, hence we are in the

case $g-1$. By our induction, we have a set of linear forms $\bar{F}(\bar{a}_{ij})$ and $\bar{G}(\bar{a}_{ij})$ which satisfy the assertion of the theorem. Now the coefficients $\bar{a}_{ij}$ of $\bar{\xi}(\bar{x},\bar{y})$ are linear homogeneous forms in the coefficients a_{ij} of $\xi(x,y)$. Let $\{F_m\}$ be the set of forms in the a_{ij} consisting of the forms $\bar{F}(\bar{a}_{ij})$, expressed in terms of the a_{ij}, and of the left-hand members of the equations (20) and (22). Let moreover, $\{G_n\}$ be the set of forms in the a_{ij} consisting of the forms $\bar{G}(\bar{a}_{ij})$ (expressed in terms of the a_{ij}) and of the form $a_{0,\nu}$. We assert that the forms $\{F_m\}$ and $\{G_n\}$ satisfy the assertion of our theorem. In the first place, by the definition of the forms F_m, the equations $F_m=0$ must be satisfied if $\xi(x,y)$ can be uniformized by an expansion $y=y_1$ of the type (18). Assume that the coefficients a_{ij} satisfy the equations $F_m=0$ and the inequalities $G_n\neq 0$. The validity of the equations (20) implies that the substitution (19) introduces in $\xi(x,y)$ a factor $\bar{x}^{\lambda}$, where $\lambda\geqq \nu\alpha'_1$. Hence in (20′) the factor $\bar{\xi}(\bar{x},\bar{y})$ contains no negative powers of $\bar{x}$ or of $\bar{y}$. The coefficients $\bar{a}_{ij}$ of $\bar{\xi}$ satisfy, by hypothesis, the equations $\bar{F}(\bar{a}_{ij})=0$ and the inequalities $\bar{G}(\bar{a}_{ij})\neq 0$, relative to the branch (21). Hence the equation $\bar{\xi}(\bar{x},\bar{y})=0$ can be uniformized by an expansion $\bar{y}=\bar{y}_1$ of type (21), and consequently also the equation $\xi(x,y)=0$ can be uniformized by an expansion $y=y_1$ of type (18). The power series $\xi(x,y)$ must be divisible by the irreducible element $\prod_{j=1}^{\nu}(y-y_1^{(j)})$. Let $\xi=\eta(x,y),\ \prod_{j=1}^{\nu}(y-y_1^{(j)})$. But, by assumption $a_{0,\nu}\neq 0$, i. e. $\xi(x,y)$ contains a term in y^{ν}. Consequently $\eta(x,y)$ must contain a constant term $\neq 0$, since the coefficient of any term y^j, $j<\nu$, in the product $\prod_{j=1}^{\nu}(y-y_1^{(j)})$, is divisible by x. Hence η is a unit, and ξ is irreducible.

Conversely, if $\xi=0$ admits a uniformization $y=y_1$ of type (18) and if ξ is irreducible, we will have $\xi=\eta(x,y)\prod_{j=1}^{\nu}(y-y_1^{(j)})$, where η is a unit. This implies in the first place $a_{0\nu}\neq 0$, and it also implies, as was pointed out before, that the equations $F_m=0$ hold true. The inequalities $G_n(a_{ij})\neq 0$ arising from the inequalities $\bar{G}(\bar{a}_{ij})\neq 0$ must also be satisfied, since the hypothesis that ξ is irreducible in $\mathfrak{k}\{x,y\}$ implies that $\bar{\xi}(\bar{x},\bar{y})$ is irreducible in $\mathfrak{k}\{\bar{x},\bar{y}\}$.

It remains to prove that the elements $\xi(x,y)=\Sigma a_{ij}x^iy^j$ whose coefficients satisfy the linear relations $F_m=0$ form a simple ideal $\mathfrak{P}$. Let us regard the coefficients a_{ij} as elements of the field defined by the equations $F_m(a_{ij})=0$. The inequalities $G_n\neq 0$ are then satisfied, and $\xi=0$ admits a uniformization $y=y_1$ of type (18). As a consequence also $x\xi=0$ and $y\xi=0$ admit the uniformization $y=y_1$, and hence the coefficients of the two power series $x\xi$

and $y\xi$ must satisfy the relations $F_m = 0$. This implies that the relations $F_m(a_{i-1,j}) = 0$, $F_m(a_{i,j-1}) = 0$ are consequences of the relations $F_m(a_{ij}) = 0$, i.e. if ξ is any element in $\mathfrak{P}$, then $x\xi \varepsilon \mathfrak{P}$ and $y\xi \varepsilon \mathfrak{P}$. Hence $\mathfrak{P}$ is an ideal. To prove that $\mathfrak{P}$ is a simple ideal, we observe, that if $\mathfrak{P}$ was a composite ideal, then, by Theorem 10. 1, a suitable quasi-general element t of $\mathfrak{P}$ would be reducible in the algebraic closure of the field of the coefficients of t. Now t, a quasi-general element of $\mathfrak{P}$, has the property that its coefficients satisfy no linear relations other than those which hold for the coefficients of the general element of $\mathfrak{P}$, i.e. only the relations $F_m = 0$. Hence the coefficients of t certainly satisfy the inequalities $G_n \neq 0$, and therefore t could not be reducible, in contradiction with our assumption that $\mathfrak{P}$ is composite, q. e. d.

In order to apply the above theorem, we begin with some preliminary remarks. Let $t = \Sigma t_{ij}x^iy^j$ be the general element of a valuation ideal $\mathfrak{q}$, and let us assume that t is absolutely irreducible (this is certainly the case if $\mathfrak{q}$ is a simple v-ideal, see Theorem 10. 2, Corollary). We will have then $t = \epsilon. \prod_{i=1}^{\nu} (y - y_i)$, where ϵ is a unit and where $y_1, y_2, \cdots, y_\nu$ are the ν determinations of a Puiseux series $y_1 = \sum_{i=1}^{\infty} t_i x^{i/\nu}$, the t_i's being algebraic functions of the t_{ij}. Some of the coefficients t_i may be constants, i.e. elements in $\mathfrak{k}$. Let $t_i = c_i$, $i = 1, 2, \cdots, h$, $c_i \varepsilon \mathfrak{k}$, while t_{h+1} is the first coefficient which is transcendental with respect to $\mathfrak{k}$. The ideal $\mathfrak{q}$ is a valuation ideal for some algebraic valuation B, defined by an expansion $y = \eta = \sum_{i=1}^{\infty} d_i x^{i/\mu}$, $d_i \varepsilon \mathfrak{k}$, and the value of t, i.e. the evaluation of $\mathfrak{q}$, is the exponent of the term of lowest degree in $t(x, \eta) = \epsilon. \prod_{i=1}^{\nu} (y - y_i)$. It is clear that the term of lowest degree in $\eta - y_i$ is the same as the term of lowest degree in $\eta - \Sigma c_i x^{i/\nu} - t_{h+1}x^{(h+1)/\nu}$. It follows that the value of t is not altered if we regard the coefficients $t_{h+1}, t_{h+2}, \cdots$ in the expansion y_1 as entirely independent indeterminates. Now if $\bar{t}(x, y)$ denotes the *direct* product $\epsilon. \prod_{i=1}^{\nu} (y - y_i)$, in which $t_{h+1}, t_{h+2}, \cdots$ are regarded as indeterminates and ϵ is a unit with indeterminate coefficients (i.e. ϵ is the general element of the unit ideal), then $\bar{t}$ is a variable element of $\mathfrak{q}$, since $v(\bar{t}) = v(\mathfrak{q})$, and on the other hand $\bar{t}$ is at least as general a power series as t (i.e. t is a specialization of $\bar{t}$). Since t is the general element of $\mathfrak{q}$, it follows that t can be identified with $\bar{t}$, and hence the coefficients $t_{h+1}, t_{h+2}, \cdots$ in the original Puiseux series y_1 are indeed algebraically independent with respect to $\mathfrak{k}$, and can be regarded as indeterminates. Changing slightly our notation and putting into evidence the coefficients c_i which are different from zero, we re-write the series y_1 as follows:

$$(23)\qquad y_1 = \sum_{i=1}^{k} c_i x^{\alpha_i/\nu} + \sum_{j=0}^{\infty} t_j x^{(\alpha_{k+1}+j)/\nu}, \qquad \begin{array}{l} 0 < \alpha_1 < \alpha_2 < \cdots, \\ c_1 c_2 \cdots c_k \neq 0, \end{array}$$

where $t_0, t_1, t_2,$ are indeterminates. Let δ be the highest common divisor of $\alpha_1, \alpha_2, \cdots, \alpha_{k+1}, \nu$. *We assert that if* $\delta > 1$, *then* $t = \epsilon \prod_{i=1}^{\nu} (y - y_i)$ *cannot be the general element of an ideal.* We shall show, namely, that if $\delta > 1$, then the coefficients t_{ij} of $t = \Sigma t_{ij} x^i y^j$ satisfy non-linear relations (which are not consequences of linear relations). Consider another sample of the series (23),

$$y'_1 = \sum_{i=1}^{k} c_i x^{\alpha_i/\nu} + \sum_{j=0}^{\infty} t'_j x^{(\alpha_{k+1}+j)/\nu},$$

where the t'_j are new indeterminates, and let $t' = \sum t'_{ij} x^i y^j = \epsilon' \prod_{i=1}^{\nu} (y - y'_i)$, where ϵ' is a unit with indeterminate coefficients. Also t' is a general element of $\mathfrak{q}$. Assume that the coefficients t_{ij} (and hence also the coefficients t'_{ij}) satisfy only linear homogeneous relations. Then it is clear that $t + t'$ is also a general element of $\mathfrak{q}$, whence $t + t' = \bar{\epsilon} \prod_{i=1}^{\nu} (y - \bar{y}_i)$, where $\bar{\epsilon}$ is a unit and

$$\bar{y}_1 = \sum_{i=1}^{k} c_i x^{\alpha_i/\nu} + \sum_{j=0}^{\infty} \tau_j x^{(\alpha_{k+1}+j)\nu}.$$

The substitution $y = \bar{y}_1$ must annihilate $t + t'$, i. e. $\epsilon \prod_{i=1}^{\nu} (y - y_i) + \epsilon' \prod_{i=1}^{\nu} (y - y'_i)$. We proceed to express the fact that the coefficients of the first two terms of $\epsilon \prod_{i=1}^{\nu} (\bar{y}_1 - y_i) + \epsilon' \prod_{i=1}^{\nu} (\bar{y}_1 - y'_i)$ vanish. If ω denotes a primitive ν-th root of unity, then

$$y_i = \sum_{j=1}^{k} c_j (\omega^i x^{1/\nu})^{\alpha_j} + t_0 (\omega^i x^{1/\nu})^{\alpha_{k+1}} + \sum_{j=1}^{\infty} t_j (\omega^i x^{1/\nu})^{\alpha_{k+1}+j},$$

whence

$$(24)\qquad \begin{aligned} \bar{y}_1 - y_i = \sum_{j=1}^{k} c_j (1 - \omega^{i\alpha_j}) x^{\alpha_j/\nu} &+ (\tau_0 - t_0 \omega^{i\alpha_{k+1}}) x^{\alpha_{k+1}/\nu} \\ &+ (\tau_1 - t_1 \omega^{i(\alpha_{k+1}+1)}) x^{(\alpha_{k+1}+1)/\nu} + \cdots. \end{aligned}$$

Let δ' be the h. c. d. of $\alpha_1, \alpha_2, \cdots, \alpha_k, \nu$, whence $\delta \equiv 0(\delta')$. We denote by $\Pi'(\bar{y}_1 - y_i)$ the product of the δ' factors $\bar{y}_1 - y_i$ for which $i\delta' \equiv 0(\nu)$, and by $\Pi''(\bar{y}_1 - y_i)$ the product of the remaining factors $\bar{y}_1 - y_i$, so that

$$\Pi'(\bar{y}_1 - y_i)\, \Pi''(\bar{y}_1 - y_i) = \prod_{i=1}^{\nu} (\bar{y}_1 - y_i).$$

If $i\delta' \not\equiv 0(\nu)$, the differences $1 - \omega^{i\alpha_j}$, $j = 1, 2, \cdots, k$, cannot all vanish.

Let σ be the smallest value of j, such that $1-\omega^{i a_\sigma} \neq 0$. Then, by (24), $c_\sigma(1-\omega^{i a_\sigma})x^{a_\sigma/\nu}$, $\sigma \leqq k$, is the term of smallest degree in $\bar{y}_1 - y_i$, while the exponent of the next term will be greater than $\alpha_\sigma/\nu + 1/\nu$, since, for $j = 1, 2, \cdots, k$, $\alpha_{j+1} - \alpha_j \equiv 0(\delta)$, whence $\alpha_{j+1} - \alpha_j \geqq \delta > 1$. It follows that

$$(25) \qquad \Pi''(\bar{y}_1 - y_i) = dx^\lambda + d_1 x^{\lambda_1} + \cdots, \ 0 \neq d \in \mathfrak{k}, \ \lambda_1 > \lambda + 1/\nu.$$

Similarly, replacing the t_j by the t'_j, we will have

$$(26) \qquad \Pi''(\bar{y}_1 - y'_i) = dx^\lambda + d'_1 x^{\lambda_1} + \cdots.$$

We now consider the product $\Pi'(\bar{y}_1 - y_i)$. Here i assumes the values $\nu/\delta', 2\nu/\delta', \cdots, \delta'\nu/\delta'$, whence

$$(27) \qquad \begin{aligned} \bar{y}_1 - y_i = \bar{y} - y_{j\nu/\delta'} &= (\tau_0 - t_0\omega_1^{j a_{k+1}})x^{a_{k+1}/\nu} \\ &+ (\tau_1 - t_1\omega_1^{j(a_{k+1}+1)})x^{(a_{k+1}+1)/\nu} + \cdots, \ \omega_1 = \omega^{\nu/\delta'}, \\ &\qquad (j = 1, 2, \cdots, \delta'), \end{aligned}$$

where ω_1 is a primitive root of unity of exponent δ'. Taking into account that $(\delta', \alpha_{k+1}) = \delta > 1$ and letting $\delta' = \delta h$, we find from (27),

$$\Pi'(\bar{y}_1 - y_i) = (\tau_0{}^h - t_0{}^h)^\delta x^{\delta' a_{k+1}/\nu} + \delta h(\tau_0{}^h - t_0{}^h)^{\delta-1}\tau_0{}^{h-1}\tau_1 x^{(\delta' a_{k+1}+1)/\nu} + \cdots.$$

It follows, by (25), that

$$\prod_{i=1}^{\nu}(\bar{y}_1 - y_i) = d(\tau_0{}^h - t_0{}^h)^\delta x^{\lambda+\delta' a_{k+1}/\nu} + d\delta h(\tau_0{}^h - t_0{}^h)^{\delta-1}\tau_0{}^{h-1}\tau_1 x^{(\delta' a_{k+1}+1)/\nu+\lambda} + \cdots.$$

The first two terms of $\prod_{i=1}^{\nu}(\bar{y}_1 - y'_i)$ are obtained from the above by replacing t_0 by t'_0. Hence we must have

$$\epsilon_{00}(\tau_0{}^h - t_0{}^h)^\delta + \epsilon'_{00}(\tau_0{}^h - t'_0{}^h)^\delta = 0, \ \epsilon_{00}(\tau_0{}^h - t_0{}^h)^{\delta-1} + \epsilon'_{00}(\tau_0{}^h - t'_0{}^h)^{\delta-1} = 0,$$

where ϵ_{00} and ϵ'_{00} are the constant terms in ϵ and ϵ' respectively. These equations imply the relations $\tau_0{}^h = t_0{}^h = t'_0{}^h$, in contradiction with the algebraic independence of the t_i and the t'_i. This proves our assertion.

This result shows that the general element $t = \epsilon \prod_{i=1}^{\nu}(y - y_i)$ of the valuation ideal $\mathfrak{q}$ is exactly of the type considered in the proof of Theorem

11. 1, and hence $\mathfrak{q}$ belongs to the class of ideals $\mathfrak{P}$ defined in that theorem. This holds true for any valuation ideal $\mathfrak{q}$ whose general element is absolutely irreducible, in particular for any simple valuation ideal. But it has been shown that the ideals $\mathfrak{P}$ are simple ideals. Hence, *a valuation ideal whose general element is irreducible is simple.* The converse has already been proved before (Theorem 10. 2, Corollary). Reassuming and recalling Theorem 10. 2, we have the following theorem:

THEOREM 11. 2. *A valuation ideal $\mathfrak{P}$ is simple if and only if its general element $t = \Sigma t_{ij}x^iy^j$ is absolutely irreducible. The general element t of $\mathfrak{P}$ is the direct product $t = \epsilon \cdot \prod_{i=1}^{\nu} (y - y_i)$, where ϵ is a unit (with indeterminate coefficients) and $y_1 = \sum_{i=1}^{k} c_i x^{\alpha_i/\nu} + \sum_{j=0}^{\infty} t_j x^{(\alpha_{k+1}+j)\nu}$, $(\alpha_1, \alpha_2, \cdots, \alpha_{k+1}, \nu) = 1$, the t_j being indeterminates. Given any valuation ideal $\mathfrak{A}$, the factorization of the general element of $\mathfrak{A}$ into irreducible factors yields a factorization of $\mathfrak{A}$ into simple v-ideals, according to the scheme indicated in Theorem* 10. 2.

We are now in position to prove the basic Theorem 5. 2 of Part I:

COROLLARY 11. 2 (Theorem 5. 2). *The transform of a simple v-ideal by a quadratic transformation is a simple v-ideal.*

Proof. It is immaterial whether the theorem is proved for polynomial ideals or for power series ideals, in view of the relationship between these ideals described in section 9. Let $\mathfrak{P}$ be a simple 0-dimensional ideal v-ideal in $\mathfrak{O}^* = \mathfrak{k}\{x, y\}$, belonging to a valuation B^*, and let $x'^{\nu}\mathfrak{P}'$ be the extended ideal of $\mathfrak{P}$ in $\mathfrak{O}'^* = \mathfrak{k}\{x', y'\}$, where $x = x'$, $y = y'x'$ (we assume as usual that $v(y) > v(x)$) and where ν is the integer such that $\mathfrak{P} \equiv 0(\mathfrak{p}^{*\nu})$, $\mathfrak{P} \not\equiv 0(\mathfrak{p}^{*\nu+1})$, $\mathfrak{p}^* = (x, y)$. Here $\mathfrak{P}'$ is necessarily either zero-dimensional or the unit ideal. Let $t(x, y) = \Sigma t_{ij}x^iy^j$ be the general element of $\mathfrak{P}$. By the definition of the integer ν, we must have $t_{ij} = 0$ for all i, j such that $i + j < \nu$, and $t_{ij} \neq 0$ for some i, j such that $i + j = \nu$. Hence $t(x, y) = t(x', x'y') = x'^{\nu}\tau(x', y')$, where $\tau(x', y') = \Sigma \tau_{ij}x'^iy'^j$ is a formal power series in x', y'. Here $\tau_{ij} = t_{i+\nu-j,j}$, if $i + \nu \geqq j$, and $\tau_{ij} = 0$, if $i + \nu < j$. In our special case the general element t of $\mathfrak{P}$ is of the form indicated in Theorem 11. 2. Hence

$$\tau = \epsilon(x, y) \prod_{i=1}^{\nu} (y' - y'_i),$$

where

$$y'_1 = \sum_{i=1}^{k} c_i x'^{(a_i - \nu)/\nu} + \sum_{j=0}^{\infty} t_j x'^{(a_{k+1}+j-\nu)/\nu}.$$

If then $\epsilon'(x', y')$ denote a general unit of $\mathfrak{O}'^*$ (i. e. a unit with indeterminate coefficients), then the direct product $\epsilon'\tau$ is the general element of a simple ideal $\bar{\mathfrak{P}}'$. We have $v(\bar{\mathfrak{P}}') = v(\tau)$ and

$$v(\tau) + v(x'^{\nu}) = v(t) = v(\mathfrak{P}),$$

whence $v(\bar{\mathfrak{P}}') = v(\mathfrak{P}')$ and $\bar{\mathfrak{P}}' \equiv 0(\mathfrak{P}')$.

On the other hand, any element $t^0 = \Sigma t_{ij}{}^0 x^i y^j$ of $\mathfrak{P}$ is obtained from the general element t by a specialization $t_{ij} \rightarrow t_{ij}{}^0$. Hence, if

$$t^0(x, y) = x'^{\nu}\tau^0(x', y') = x'^{\nu}\Sigma\tau_{ij}{}^{(0)} x'^i y'^j,$$

then τ^0 is obtained from $\epsilon'\tau$ by the specialization $\epsilon' \rightarrow 1$, $\tau_{ij} \rightarrow \tau_{ij}{}^0$, and therefore τ^0 is an element of the ideal $\bar{\mathfrak{P}}'$. Since $x'^{\nu}\mathfrak{P}'$ is the extended ideal of $\mathfrak{P}$, a finite number of elements such as τ^0 form a base of $\mathfrak{P}'$. Hence $\mathfrak{P}' \equiv 0(\bar{\mathfrak{P}}')$, and consequently $\mathfrak{P}' = \bar{\mathfrak{P}}'$, q. e. d.

Remark 1. In view of the unicity of the factorization of a v-ideal $\mathfrak{A}$ into simple v-ideals (Theorem 7. 1), the second part of Theorem 11. 2 implies that the factorization of the general element of $\mathfrak{A}$ into irreducible factors yields *the* factorization of $\mathfrak{A}$ into simple v-ideals.

Remark 2. Concerning the characterization of simple v-ideals given in Theorem 11. 2, it is not difficult to show that, conversely, *an ideal $\mathfrak{P}$ whose general element t is of the type described in Theorem* 11. 2, *is a v-ideal* (necessarily simple). For the proof we assume $\alpha_1 > \nu$ and we consider the transform $\mathfrak{P}'$ of $\mathfrak{P}$ by the quadratic transformation $x' = x$, $y' = y/x$: $\mathfrak{O}'^*\mathfrak{P} = x'^{\nu}\mathfrak{P}'$. Using the notation of the proof above, we find as above the congruence $\mathfrak{P}' \equiv 0(\bar{\mathfrak{P}}')$. The preceding proof of the congruence $\bar{\mathfrak{P}}' \equiv 0(\mathfrak{P}')$ was based upon the fact that $\mathfrak{P}'$ was a v-ideal. But we may proceed without making use of this property of $\mathfrak{P}'$. Let $t'(x', y')$ be the general element of $\mathfrak{P}'$, $t' = \Sigma t'_{ij} x'^i y'^j$, and let $\Sigma c_{ij} t'_{ij} = 0$ be a true linear relation between the t'_{ij}. If $t^{(0)} = \Sigma t_{ij}{}^{(0)} x^i y^j$ is any element in $\mathfrak{P}$, and if $t^{(0)} = x'^{\nu}\tau_0$, $\tau_0 = \Sigma\tau_{ij}{}^{(0)} x'^i y'^j$, then τ_0 is an element of $\mathfrak{P}'$. Hence we must have

$$\Sigma\, c_{ij}\tau_{ij}{}^{(0)} = \sum_{i+\nu \geqq j} c_{ij} t^{(0)}{}_{i+\nu-j,j} = 0.$$

This relation holds for any element $t^{(0)}$ in $\mathfrak{P}$, consequently we must have $\sum_{i+\nu\geqq j} c_{ij}t_{i+\nu-j,j}=0$, whence $\sum c_{ij}\tau_{ij}=0$. This last relation shows that τ *is a variable element of* $\mathfrak{P}'$. But then, in view of Theorem 10. 1, also $\epsilon'\tau$ is a variable element of $\mathfrak{P}'$, since ϵ' is the general element of the unit ideal. Consequently $\bar{\mathfrak{P}}'\equiv 0(\mathfrak{P}')$, whence again $\mathfrak{P}'=\bar{\mathfrak{P}}'$.

Consider the valuation B defined by the branch

$$y=\zeta=\sum_{i=1}^{k} c_i x^{\alpha_i/\nu}+\sum_{j=0}^{\infty} d_j x^{(\alpha_{k+1}+j)/\nu},$$

where $d_0, d_1, \cdots$ are arbitrary constants (in $\mathfrak{k}$) and $d_0\neq 0$. Let the value of t in this valuation be λ_0/ν, λ_0—an integer. The integer $\lambda_0=\lambda(\mathfrak{P})$ is independent of the particular constants d_j and is uniquely determined by the ideal $\mathfrak{P}$ and by the auxiliary condition $\alpha_1\geqq\nu$ which prevents a special position of the axis $x=0$. If the variables x', y' are used, then the valuation B is defined by the branch

$$y'=\zeta'=\sum_{i=1}^{k} c_i x'^{(\alpha_i-\nu)/\nu}+\sum_{j=0}^{\infty} d_j x^{(\alpha_{k+1}+j-\nu)/\nu}.$$

We have $v(\mathfrak{P}')=v(\mathfrak{P})-v(x^\nu)$, whence $v(\mathfrak{P}')=(\lambda_0-\nu^2)/\nu$. If $\alpha_1-\nu\geqq\nu$, then $\lambda(\mathfrak{P}')=\lambda_0-\nu^2<\lambda(\mathfrak{P})$. If $\alpha_1-\nu<\nu$, then the rôles of the variables x', y' must be interchanged, and putting $\alpha_1-\nu=\nu'$ we find $v(\mathfrak{P}')=[(\lambda_0-\nu^2)/\nu]\cdot(\nu/\nu')$, whence again $\lambda(\mathfrak{P}')=\lambda_0-\nu^2<\lambda(\mathfrak{P})$. The inequality $\lambda(\mathfrak{P}')<\lambda(\mathfrak{P})$ leads to a complete induction with respect to $\lambda(\mathfrak{P})$. If $\lambda(\mathfrak{P})=1$, then $\nu=1$, since evidently $\lambda(\mathfrak{P})\geqq\nu$, and hence $\mathfrak{P}=\mathfrak{p}^*=(x,y)$, so that if $\lambda(\mathfrak{P})=1$, $\mathfrak{P}$ is a v-ideal. Since $\lambda(\mathfrak{P}')<\lambda(\mathfrak{P})$ we may assume, according to our induction, that $\mathfrak{P}'$ is a v-ideal. Now $x'^\nu\mathfrak{P}'$ is the extended ideal of $\mathfrak{P}$, and from the expression of the general element t of $\mathfrak{P}$ it is seen that the subform of degree ν of t contains only the term y^ν (since $\alpha_1>\nu$). By Theorem 4. 3, our assertion that $\mathfrak{P}$ is a v-ideal, will follow, provided it is shown that $\mathfrak{P}$ *is the contracted ideal of* $x'^\nu\mathfrak{P}'$. The proof of this is immediate. The general element $\bar{t}$ of the contracted ideal of $x'^\nu\mathfrak{P}'$ is of the form $x'^\nu\bar{\tau}(x',y')$, and we can assert that not only is $\bar{\tau}$ a variable element of $\mathfrak{P}'$ (this is obvious) but also that its coefficients $\bar{\tau}_{ij}$ satisfy those inequalities (see Theorem 11. 1) which insure the irreducibility of $\bar{\tau}$, *because this is true for the general element* $t=x'^\nu\tau(x',y')$ *of the ideal* $\mathfrak{P}$. It follows that $\bar{t}$ is necessarily of the form $\bar{t}=\epsilon(x,y)\prod_{i=1}^{\nu}(y-y_i)$, where $y_i=\sum c_i x^{\alpha_i/\nu}+\sum_{j=0}^{\infty} t_j x^{(\alpha_{k+1}+j)/\nu}$, whence $\bar{t}$ is a variable element of $\mathfrak{P}$. Since $\mathfrak{P}$ is contained in the contracted ideal of $x'^\nu\mathfrak{P}'$, it follows that $\mathfrak{P}$ coincides with the contracted ideal, q. e. d.

12. The class of complete ideals. It has been pointed out in section 7 that a product of valuation ideals is not always a valuation ideal. The class of ideals (in $\mathfrak{k}[x,y]$ or in $\mathfrak{k}\{x,y\}$) which can be factored into valuation ideals is therefore larger than the class of valuation ideals. We shall call ideals in this class *complete ideals.* The term "complete" is suggested by the notion of complete linear systems in algebraic geometry, these being the linear systems which are defined uniquely by *base conditions,* i. e. by the condition of passing with assigned multiplicities through an assigned set of proper or infinitely near points. It will be seen that complete ideals are those and only those ideals whose elements are subject to given base conditions, and to no other conditions. In other words, the polynomials which belong to a complete ideal and whose degree is not greater than a given integer n, form, for any n, a complete linear system.

By our definition of complete ideals, the class of complete ideals is closed under multiplication. Moreover, by Theorem 7. 1, a complete ideal has a unique factorization into simple complete ideals, these last ones being necessarily valuation ideals. Our next aim is to prove that this class is also closed under the other ideal operations ([,]), (:) (intersection and quotient); not however under addition (+).* The theorem which we wish to prove is the following:

THEOREM 12. 1. *If $\mathfrak{A}$ and $\mathfrak{B}$ are complete ideals and $\mathfrak{C}$ is an arbitrary ideal, then $[\mathfrak{A},\mathfrak{B}]$ and $\mathfrak{A}:\mathfrak{C}$ are complete ideals.*

This theorem is an immediate consequence of the following:

LEMMA. *Any complete ideal is the intersection of valuation ideals, and, conversely, the intersection of valuation ideals is a complete ideal.*

Proof of the lemma. Let $\mathfrak{A}=[\mathfrak{A}_1,\mathfrak{A}_2,\cdots,\mathfrak{A}_k]$, where each $\mathfrak{A}_i$ is a valuation ideal for some valuation B_i. We may assume that the $\mathfrak{A}_i$ belong to one and the same prime ideal $\mathfrak{p}=(x,y)$. Denote by t the general element of $\mathfrak{A}$ and let $t=t_1t_2\cdots t_m$ be the factorization of t into irreducible factors in the ring $\mathfrak{K}\{x,y\}$, where $\mathfrak{K}$ is the algebraic closure of the field of the coefficients of t. Let $\alpha_{ij}=v(t_i)$ be the value of t_i in the valuation B_j $(i=1,2,\cdots,m;\ j=1,2,\cdots,k)$, and let $\mathfrak{A}_{ij}$ be the v-ideal belonging to B_j such that $v(\mathfrak{A}_{ij})=\alpha_{ij}$. Since $v(t_i)=v(\mathfrak{A}_{ij})$ in B_j, t_i is a variable element of $\mathfrak{A}_{ij}$, whence t_i is also a variable element of the intersection

* *Example.* The ideals $(y^2,\mathfrak{p}^3)$, $(x^2,\mathfrak{p}^3)$ are v-ideals, but their sum (join) is the *simple* ideal $(x^2,y^2,\mathfrak{p}^3)$, which is obviously not a v-ideal and consequently not complete.

$$\mathfrak{B}_i = [\mathfrak{A}_{i1}, \mathfrak{A}_{i2}, \cdots, \mathfrak{A}_{ik}], \qquad (i = 1, 2, \cdots, m).$$

Let τ_i be the general element of $\mathfrak{B}_i$. We have $v(\tau_i) \leqq v(t_i)$ in B_j, since all the linear relations between the coefficients of τ_i are also satisfied by the coefficients of t_i. On the other hand, we have in any of the valuations B_j, $v(\tau_i) = v(\mathfrak{B}_i) \geqq v(\mathfrak{A}_{ij}) = v(t_i)$. Hence $v(\tau_i) = v(t_i)$. Let $\tau = \tau_1\tau_2 \cdots \tau_m$ be the direct product of the elements τ_i. The elements t and τ have the same value in each of the valuations B_j, whence $v(\tau) = v(\mathfrak{A}) \geqq v(\mathfrak{A}_j)$ in B_j. This shows that τ is a variable element of $\mathfrak{A}_j$, whence τ also is a variable element of the intersection $\mathfrak{A}$ of the $\mathfrak{A}_j$. But τ is not less general than t (since τ_i is the general element of the ideal $\mathfrak{B}_i$, of which t_i is a variable element, and since τ is a *direct* product of the τ_i); consequently, since t is the general element of $\mathfrak{A}$, also τ is the general element of $\mathfrak{A}$. We may then identify t with τ, and we deduce *that t is the direct product of its irreducible factors t_i and that t_i is the general element of $\mathfrak{B}_i$.*

We now apply the considerations developed in the proof of Theorem 11.2. For the irreducible element t_i we have the following uniformization: $t_i = \epsilon_i(x, y) \prod_{h=1}^{\nu_i} (y - y_h^{(i)})$, where

$$y_1^{(i)} = \sum_{j=1}^{k} c_{ij} x^{\alpha_{ij}/\nu_i} + \sum_{j=0}^{\infty} \tau_{ij} x^{(\alpha_{i,k+1}+j)/\nu_i},$$

τ_{i0}—transcendental with respect to field $\mathfrak{k}$.

Since the inequalities $v(t) \geqq v(\mathfrak{A}_j)$ in B_j, $j = 1, 2, \cdots, k$, are the only conditions which should be imposed on t, it follows that all the τ_{ij} are indeterminates, as the value of t_i in any algebraic valuation is not altered if we regard the τ_{ij} as indeterminates. Also $\epsilon_i(x, y)$ is to be regarded as a unit element with indeterminate coefficients, and the product of ϵ_i by $\prod_{i=1}^{\nu_i} (y - y_h^{(i)})$ is to be intended as direct. But then we must have necessarily $(\alpha_{i1}, \alpha_{i2}, \cdots, \alpha_{i,k+1}, \nu_i) = 1$ and consequently t_i is the general element of a simple v-ideal (by the preceding section, Remark 2). Hence $\mathfrak{B}_i = \mathfrak{P}^{(i)}$, where $\mathfrak{P}^{(i)}$ is a simple v-ideal. By Theorem 10.1, $t_1t_2 \cdots t_m$ is a quasi-general element of the ideal $\mathfrak{P}^{(1)}\mathfrak{P}^{(2)} \cdots \mathfrak{P}^{(m)}$. But the coefficients of the direct product Πt_i, i. e. of t, satisfy only linear relations, since t is the general element of $\mathfrak{A}$. Hence t is also the general element of $\mathfrak{P}^{(1)}\mathfrak{P}^{(2)} \cdots \mathfrak{P}^{(m)}$, and consequently $\mathfrak{A} = \mathfrak{P}^{(1)}\mathfrak{P}^{(2)} \cdots \mathfrak{P}^{(m)}$. This proves the first part of the lemma.

To prove the second part of the lemma, let $\mathfrak{A} = \mathfrak{A}_1\mathfrak{A}_2 \cdots \mathfrak{A}_m$ be a complete ideal, where $\mathfrak{A}_1, \mathfrak{A}_2, \cdots, \mathfrak{A}_m$ are simple v-ideals, which we may assume as primary ideals belonging to one and the same prime ideal $\mathfrak{p} = (x, y)$. Let

$\mathfrak{A}_j$ be of kind h_j, and let $h = \max(h_1, h_2, \cdots, h_m)$. The assertion of the lemma is trivial if $m = 1$. We assume that the assertion is true for all integers h' such that $h' < h$ (if $h = 1$, then necessarily $m = 1$). Let $\mathfrak{A}_j$ belong to a valuation B_j. Assuming that $v(y) \geqq v(x)$ in each of the valuation B_j, we apply the quadratic transformation T, $x' = x$, $y' = y/x$ to the polynomial ring $\mathfrak{O} = \mathfrak{k}[x, y]$, getting the ring $\mathfrak{O} = \mathfrak{k}[x', y']$ of polynomials in x', y'. Let $\mathfrak{O}'\mathfrak{A}_j = x^{\rho_j}\mathfrak{A}'_j$, whence $\mathfrak{O}'\mathfrak{A} = x'^{\rho}\mathfrak{A}'_1\mathfrak{A}'_2 \cdots \mathfrak{A}'_m$, $\rho = \sum_{j=1}^{m} \rho_j$, where $\mathfrak{A}_j \equiv 0(\mathfrak{p}^{\rho_j})$ and $\mathfrak{A}_j \not\equiv 0(\mathfrak{p}^{\rho_j+1})$. By Theorem 4. 1, $\mathfrak{A}'_j$ is a v-ideal in $\mathfrak{O}'$, and by Theorems 5. 2 and 5. 3 $\mathfrak{A}'_j$ is a simple v-ideal of kind $h_j - 1$. The product $\prod_{i-1}^{\rho} \mathfrak{A}'_j$ can be first written as the intersection of the partial products consisting of factors $\mathfrak{A}'_j$ which belong to one and the same prime ideal. Then, by our induction, each partial product can be written as the intersection of v-ideals. Let then

$$\mathfrak{A}' = \mathfrak{A}'_1\mathfrak{A}'_2 \cdots \mathfrak{A}'_m = [\mathfrak{B}'_1, \mathfrak{B}'_2, \cdots, \mathfrak{B}'_k],$$

whence

$$\mathfrak{O}'\mathfrak{A} = [x^{\rho}\mathfrak{B}'_1, x^{\rho}\mathfrak{B}'_2, \cdots, x^{\rho}\mathfrak{B}'_k], \tag{28}$$

where $\mathfrak{B}'_1, \mathfrak{B}'_2, \cdots, \mathfrak{B}'_k$ are v-ideals in $\mathfrak{O}'$.

Let σ_i be the smallest integer such that $x'^{\sigma_i}\mathfrak{B}'_i$ is an extended ideal of an ideal in $\mathfrak{O}$, and let $\mathfrak{B}_i$ be the contracted ideal of $x'^{\sigma_i}\mathfrak{B}'_i$. By Theorem 4. 3, $\mathfrak{B}_i$ is a v-ideal and $\mathfrak{B}_i = T^{-1}(\mathfrak{B}'_i)$. We have $\mathfrak{O}'\mathfrak{B}_i = x'^{\sigma_i}\mathfrak{B}'_i$ and $\mathfrak{O}'\mathfrak{A} = x^{\rho}\mathfrak{A}'$. We assert that $\sigma_i \leqq \rho$. In fact, assuming that $\sigma_i > \rho$, we have

$$\mathfrak{O}'\mathfrak{p}^{\sigma_i - \rho}\mathfrak{A} = x'^{\sigma_i}\mathfrak{A}' \subseteqq x'^{\sigma_i}\mathfrak{B}'_i,$$

whence, passing to the contracted ideal of $x'^{\sigma_i}\mathfrak{B}'_i$, $\mathfrak{p}^{\sigma_i-\rho}\mathfrak{A} \equiv 0(\mathfrak{B}_i)$. Now $\mathfrak{A} \equiv 0(\mathfrak{p}^{\rho})$ and $\mathfrak{A} \not\equiv 0(\mathfrak{p}^{\rho+1})$, hence $\mathfrak{p}^{\sigma_i-\rho}\mathfrak{A} \equiv 0(\mathfrak{p}^{\sigma_i})$ and $\not\equiv 0(\mathfrak{p}^{\sigma_i+1})$. Since also $\mathfrak{B}_i$ is in $\mathfrak{p}^{\sigma_i}$ and not in $\mathfrak{p}^{\sigma_i+1}$, the congruence $\mathfrak{p}^{\sigma_i-\rho}\mathfrak{A} \equiv 0(\mathfrak{B}_i)$ implies that the subforms of degree σ_i of the polynomials in $\mathfrak{B}_i$ form a linear system $\Omega(\mathfrak{B}_i)$ of dimension $\geqq \sigma_i - \rho > 0$. This is impossible since, by our definition of $\mathfrak{B}_i(= T^{-1}(\mathfrak{B}'_i))$, the system $\Omega(\mathfrak{B}_i)$ is of dimension zero.

We have therefore $\sigma_i \leqq \rho$, and consequently $x^{\rho}\mathfrak{B}'_i$ is an extended ideal, its contracted ideal being $\mathfrak{p}^{\rho-\sigma_i}\mathfrak{B}_i$ (Theorem 4. 3). Denoting by $\mathfrak{A}^*$ the contracted ideal of $\mathfrak{O}'\mathfrak{A}$, we obtain from (28):

$$\mathfrak{A} \subseteqq \mathfrak{A}^* = [\mathfrak{p}^{\rho-\sigma_1}\mathfrak{B}_1, \mathfrak{p}^{\rho-\sigma_2}\mathfrak{B}_2, \cdots, \mathfrak{p}^{\rho-\sigma_k}\mathfrak{B}_k].$$

Now $\mathfrak{B}_i$ is a valuation ideal for some valuation, and in that valuation the

ideal $\mathfrak{p}^{\rho-\sigma_i}\mathfrak{B}_i$ is equivalent to some valuation ideal $\mathfrak{C}_i$. By Corollary 3. 3 we have $\mathfrak{p}^{\rho-\sigma_i}\mathfrak{B}_i = [\mathfrak{C}_i, \mathfrak{p}^\rho]$, whence

$$\mathfrak{A} \subseteqq \mathfrak{A}^* = [\mathfrak{C}_1, \mathfrak{C}_2, \cdots, \mathfrak{C}_k, \mathfrak{p}^\rho].$$

Hence $\mathfrak{A}^*$ is the intersection of valuation ideals.[13] By the first part of the lemma, just proved, $\mathfrak{A}^*$ is a complete ideal,

$$\mathfrak{A}^* = \mathfrak{A}^*_1\mathfrak{A}^*_2 \cdots \mathfrak{A}^*_n,$$

where the $\mathfrak{A}^*_j$ are simple v-ideals. $\mathfrak{A}$ and $\mathfrak{A}^*$ have the same extended ideal $x'^\rho\mathfrak{A}'$. If then $T(\mathfrak{A}^*_j) = \mathfrak{A}'^*_j$ we must have

$$x'^\rho\mathfrak{A}'_1\mathfrak{A}'_2 \cdots \mathfrak{A}'_m = \mathfrak{O}'\mathfrak{A} = \mathfrak{O}'\mathfrak{A}^* = x'^\rho\mathfrak{A}'^*_1\mathfrak{A}'^*_2 \cdots \mathfrak{A}'^*_n.$$

By the unique factorization theorem (7. 1), it follows, for a proper ordering of the factors, $m = n$, $\mathfrak{A}'_j = \mathfrak{A}'^*_j$, whence also $\mathfrak{A}_j = \mathfrak{A}^*_j$, $\mathfrak{A} = \mathfrak{A}^*$, and consequently $\mathfrak{A}$ is the intersection of v-ideals, q. e. d.

Theorem 12. 1 now follows immediately. That $[\mathfrak{A}, \mathfrak{B}]$ is a complete ideal, if $\mathfrak{A}$ and $\mathfrak{B}$ are complete ideals, is now trivial, since, by our lemma, complete ideals can also be defined as intersections of valuation ideals. Consider now the ideal $\mathfrak{A} : \mathfrak{C}$, where $\mathfrak{A}$ is a complete ideal and $\mathfrak{C}$ is an arbitrary ideal. We have $\mathfrak{A} = [\mathfrak{A}_1, \mathfrak{A}_2, \cdots, \mathfrak{A}_k]$, where the $\mathfrak{A}_j$ are valuation ideals, whence $\mathfrak{A} : \mathfrak{C} = [\mathfrak{A}_1 : \mathfrak{C}, \mathfrak{A}_2 : \mathfrak{C}, \cdots, \mathfrak{A}_k : \mathfrak{C}]$. But by Theorem 2. 1, the quotient $\mathfrak{A}_j : \mathfrak{C}$ is again a valuation ideal. Hence $\mathfrak{A} : \mathfrak{C}$ is the intersection of valuation ideals, and therefore is a complete ideal, q. e. d.

Corollary 12. 2. *If $\mathfrak{A}$ is a complete ideal* (in $\mathfrak{k}[x, y]$ or in $\mathfrak{k}\{x, y\}$) *and if $x'^\rho\mathfrak{A}'$ is the extended ideal $\mathfrak{O}'\mathfrak{A}$ in the ring $\mathfrak{k}[x', y']$ or $\mathfrak{k}\{x', y'\}$, where $x' = x$, $y' = y/x$, then $\mathfrak{A}'$ is a complete ideal and $\mathfrak{A}$ is the contracted ideal of $x'^\rho\mathfrak{A}'$.*

That $\mathfrak{A}'$ is a complete ideal is trivial. The second part of the corollary follows from the relation $\mathfrak{A} = \mathfrak{A}^*$ established in the course of the proof of the second part of the lemma.

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[13] *Any power of $\mathfrak{p}$ is a valuation ideal.* Consider the 0-dimensional valuation defined by the divisor (x') and by the point $y' = 0$ of this divisor, i. e. if $F(x', y')$ is any rational function of x', y', then put $v(F) = (m, n)$, where x'^m is the highest power of x' which divides F, $F = x'^m F_1(x'y')$, $F_1(0, y') \neq 0$, and where y'^n is the highest power of y' which divides $F_1(0, y')$. Then it is easily seen that the sequence of v-ideals

When a linear system of curves $f = 0$ is subjected to base conditions, it is required that the curves of the system have assigned intersection multiplicities with an assigned set of algebraic branches γ_j. For each branch γ_j the corresponding condition is equivalent to the condition that the value of the polynomial f in the valuation B_j defined by the branch γ_j be not inferior to a given integer; in other words: it is required that $f \equiv 0(\mathfrak{A}_j)$, where $\mathfrak{A}_j$ is a given valuation ideal belonging to B_j. The full set of base conditions is then described by the congruence: $f \equiv 0([\mathfrak{A}_1, \mathfrak{A}_2, \cdots])$, i. e. by the condition that f belong to a given complete ideal. However, the representation of a complete ideal as an intersection of valuation ideals is not unique. For instance, let $\mathfrak{A} = (xy, \mathfrak{p}^3)$, where $\mathfrak{p} = (x, y)$. $\mathfrak{A}$ is not a valuation ideal, since it has only one subform xy and this is not a power of a linear form. Let $\mathfrak{A}_1 = (x^2, xy, \mathfrak{p}^3)$, $\mathfrak{A}_2 = (xy, y^2, \mathfrak{p}^3)$, $\mathfrak{A}'_1 = (x, x^2, xy, \mathfrak{p}^3)$, $\mathfrak{A}'_2 = (y, xy, y^2, \mathfrak{p}^3)$. These 4 ideals are valuation ideals and we have $\mathfrak{A} = [\mathfrak{A}_1, \mathfrak{A}_2] = [\mathfrak{A}'_1, \mathfrak{A}'_2]$. This ambiguity is the algebraic equivalent of the distinction which the geometric theory makes between the *assigned* or *virtual* multiplicities of the curves of a linear system and the *effective* multiplicities of these curves. It is the representation of a complete ideal as a *product* of valuation ideals that is unique and puts into evidence the effective multiplicities of the general curve of a linear system.*

We point out explicitly the following result arrived at in the course of the first part of the above proof: *the general element t of a complete ideal $\mathfrak{A}$ is the direct product of its irreducible factors, and the factorization of t yields the factorization of $\mathfrak{A}$ into simple v-ideals.* This is the generalization to complete ideals of the similar property of valuation ideals given in Theorem 11. 2.

We conclude this section with the definition of an operation which assigns to each ideal $\mathfrak{A}$ in $\mathfrak{O}$ a uniquely determined complete ideal $\mathfrak{A}'$, *the complete ideal determined by* $\mathfrak{A}$. The analogue of this operation in the theory of linear systems is given by the passage from an arbitrary linear system to the corresponding complete linear system. *We define $\mathfrak{A}'$ as the intersection of all the complete ideals containing* $\mathfrak{A}$. Since $\mathfrak{A}$ has a finite length, it is clear that $\mathfrak{A}'$ is also the intersection of a finite number of complete ideals, *hence $\mathfrak{A}'$ itself is a complete ideal*, i. e. *$\mathfrak{A}'$ is the smallest complete ideal containing* $\mathfrak{A}$. The operation comes under the heading of the (′) *operations* studied in other connections by van der Waerden ([9], § 103), Prüfer [8] and others, since it enjoys the following formal properties:

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in $\mathfrak{k}[x, y]$ consists of all ideals of the form $(x^{\lambda}y^{\rho-\lambda}, x^{\lambda-1}y^{\rho-\lambda+1}, \ldots, y^{\rho}, \mathfrak{p}^{\rho+1})$, $\rho \geqq \lambda \geqq 0$, $\rho = 1, 2, 3, \cdots$. For $\lambda = \rho$, we find the ideal $\mathfrak{p}^{\rho}$.

* See Note at end of Section 12.

1. $(\mathfrak{A}')' = \mathfrak{A}'$. 2. If $\mathfrak{A}_1 \supseteqq \mathfrak{A}_2$, then $\mathfrak{A}'_1 \supseteqq \mathfrak{A}'_2$.
3. $(\mathfrak{A}'_1, \mathfrak{A}'_2)' = (\mathfrak{A}_1, \mathfrak{A}_2)'$. 4. $(\mathfrak{A}'_1\mathfrak{A}'_2)' = (\mathfrak{A}_1\mathfrak{A}_2)'$.
5. $(a)' = (a)$; $((a) \cdot \mathfrak{A})' = (a) \cdot \mathfrak{A}'$.

Proof. 1. Trivial, because $\mathfrak{A}'$ is a complete ideal itself;

2. Self-evident;

3. By 2, $(\mathfrak{A}_1, \mathfrak{A}_2)' \supseteqq (\mathfrak{A}'_1, \mathfrak{A}'_2)$, whence $(\mathfrak{A}_1, \mathfrak{A}_2)' \supseteqq (\mathfrak{A}'_1, \mathfrak{A}'_2)'$. On the other hand $\mathfrak{A}'_1 \supseteqq \mathfrak{A}_1$, $\mathfrak{A}'_2 \supseteqq \mathfrak{A}_2$, whence $(\mathfrak{A}'_1, \mathfrak{A}'_2)' \supseteqq (\mathfrak{A}_1, \mathfrak{A}_2)'$, by 2.

4. By 2, we have $(\mathfrak{A}'_1\mathfrak{A}'_2)' \supseteqq (\mathfrak{A}_1\mathfrak{A}_2)'$. Let $\mathfrak{B}$ be any valuation ideal belonging to some valuation B and containing $\mathfrak{A}_1\mathfrak{A}_2$, and let $\mathfrak{B}_i$, $i = 1, 2$, be the valuation ideal for B which is equivalent to $\mathfrak{A}_i$. We have $v(\mathfrak{B}_1\mathfrak{B}_2) = v(\mathfrak{A}_1\mathfrak{A}_2) \geqq v(\mathfrak{B})$, whence $\mathfrak{B}_1\mathfrak{B}_2 \equiv 0(\mathfrak{B})$. Since $\mathfrak{A}_i \equiv 0(\mathfrak{B}_i)$ and $\mathfrak{B}_i$ is a valuation ideal, hence complete, we have $\mathfrak{A}'_i \equiv 0(\mathfrak{B}_i)$, consequently $\mathfrak{A}'_1\mathfrak{A}'_2 \equiv 0(\mathfrak{B}_1\mathfrak{B}_2)$. Since every complete ideal is the intersection of valuation ideals, it follows that $(\mathfrak{A}_1\mathfrak{A}_2)'$ is the intersection of all the valuation ideals $\mathfrak{B}$ containing $\mathfrak{A}_1\mathfrak{A}_2$. Hence $\mathfrak{A}'_1\mathfrak{A}'_2 \subseteqq (\mathfrak{A}_1\mathfrak{A}_2)'$. As a consequence also $(\mathfrak{A}'_1\mathfrak{A}'_2)' \subseteqq ((\mathfrak{A}_1\mathfrak{A}_2)')$, whence 4 follows.

We observe, however, that $(\mathfrak{A}'_1\mathfrak{A}'_2)' = \mathfrak{A}'_1\mathfrak{A}'_2$, since the product of the complete ideals $\mathfrak{A}'_1, \mathfrak{A}'_2$ is itself complete. Hence 4 can be written as follows:

4′. $\mathfrak{A}'_1\mathfrak{A}'_2 = (\mathfrak{A}_1\mathfrak{A}_2)'$.

5. A principal ideal (a) is itself a complete ideal. If $a = a_1^{\rho_1}a_2^{\rho_2} \cdots a_k^{\rho_k}$, where $a_i(x, y)$ is an irreducible polynomial, then $(a) = (a_1^{\rho_1}) \cap (a_2^{\rho_2}) \cap \cdots \cap (a_k^{\rho_k})$, and evidently $(a_i^{\rho_i})$ is a v-ideal, belonging to the 1-dimensional valuation defined by the divisor (a_i). Hence $(a)' = (a)$. In a similar straightforward manner the relation $((a) \cdot \mathfrak{A})' = (a)\mathfrak{A}'$ can be proved.

THEOREM 12.3 (*invariance of the operation* (′) *under quadratic transformations*). *If* $\mathfrak{p} = (x, y)$ *and if* $\mathfrak{A} \equiv 0(\mathfrak{p}^\rho)$, $\mathfrak{A} \not\equiv 0(\mathfrak{p}^{\rho+1})$, *then also* $\mathfrak{A}' \equiv 0(\mathfrak{p}^\rho)$, $\mathfrak{A}' \not\equiv 0(\mathfrak{p}^{\rho+1})$. *Moreover, if* $\mathfrak{B}$ *is the transform of* $\mathfrak{A}$ *under the quadratic transformation* T, *then* $\mathfrak{B}'$ *is the transform of* $\mathfrak{A}'$ *under* T, i. e. $\mathfrak{O}'\mathfrak{A} = x'^\rho\mathfrak{B}$ implies $\mathfrak{O}'\mathfrak{A}' = x'^\rho\mathfrak{B}'$.

Proof. Since any power of $\mathfrak{p}$ is a complete ideal (even a valuation ideal) and since $\mathfrak{A}'$ is the smallest complete ideal containing $\mathfrak{A}$, the congruence $\mathfrak{A} \equiv 0(\mathfrak{p}^\rho)$ implies the congruence $\mathfrak{A}' \equiv 0(\mathfrak{p}^\rho)$. Moreover, if $\mathfrak{A} \not\equiv 0(\mathfrak{p}^{\rho+1})$,

also $\mathfrak{A}' \not\equiv 0(\mathfrak{p}^{\rho+1})$ since $\mathfrak{A} \equiv 0(\mathfrak{A}')$. Let then $\mathfrak{O}'\mathfrak{A} = x'^{\rho}\mathfrak{B}$, $\mathfrak{O}'\mathfrak{A}' = x'^{\rho}\mathfrak{B}^*$, where $\mathfrak{O}' = \mathfrak{k}[x', y']$ (or $\mathfrak{O}' = \mathfrak{k}\{x', y'\}$), $x' = x$, $y' = y/x$, and where $\mathfrak{B}$ and $\mathfrak{B}^*$ are not divisible by x'. Since $\mathfrak{A} \equiv 0(\mathfrak{A}')$, also $\mathfrak{B} \equiv 0(\mathfrak{B}^*)$, and by Corollary 12. 2, $\mathfrak{B}^*$ is a complete ideal. To prove that $\mathfrak{B}^* = \mathfrak{B}'$, we have to show that if $\mathfrak{B}_1$ is any complete ideal containing $\mathfrak{B}$, then $\mathfrak{B}^* \equiv 0(\mathfrak{B}_1)$. Let σ be the smallest integer such that $x'^{\sigma}\mathfrak{B}_1$ is an extended ideal of an ideal in $\mathfrak{O}$. The same reasoning employed in the proof of the second part of the Lemma, for the derivation of the inequality $\sigma_i \leqq \rho$, can be used also now in order to show that $\sigma \leqq \rho$. It is only necessary to observe that by Corollary 12. 2 the contracted ideal of $x'^{\sigma}\mathfrak{B}_1$ is a complete ideal, say $\mathfrak{C}$. The system $\Omega(\mathfrak{C})$ of the subforms (of degree σ) of $\mathfrak{C}$ can be of dimension greater than zero, only if in the factorization of $\mathfrak{C}$ into simple v-ideals there occurs the factor $\mathfrak{p}$. We would have then $\mathfrak{C} = \mathfrak{p}\mathfrak{D}$, whence $\mathfrak{O}'\mathfrak{D} = x'^{\sigma-1}\mathfrak{B}_1$, in contradiction with the definition of the integer σ. Hence $\Omega(\mathfrak{C})$ is of dimension zero, and it was this value of the dimension of $\Omega(\mathfrak{B}_i)$ that played a rôle in the proof of the Lemma.

Since $\sigma \leqq \rho$ and $\mathfrak{B}_1 \supseteqq \mathfrak{B}$, we have $x'^{\rho}\mathfrak{B}_1 \supseteqq x'^{\rho}\mathfrak{B}$. Now let $\mathfrak{C}$ be the contracted ideal of $x'^{\sigma}\mathfrak{B}_1$. $\mathfrak{C}$ is a complete ideal, and its extension ideal is $x'^{\sigma}\mathfrak{B}_1$. Hence the extended ideal of $\mathfrak{p}^{\rho-\sigma}\mathfrak{C}$ is $x'^{\rho}\mathfrak{B}_1$, and consequently (Corollary 12. 2) $\mathfrak{p}^{\rho-\sigma}\mathfrak{C}$ is the contracted ideal of $x'^{\rho}\mathfrak{B}_1$, since $\mathfrak{p}^{\rho-\sigma}$ is also a complete ideal. Since the contracted ideal of $x'^{\rho}\mathfrak{B}$ contains $\mathfrak{A}$, the congruence $x'^{\rho}\mathfrak{B}_1 \supseteqq x'^{\rho}\mathfrak{B}$ implies the congruence $\mathfrak{p}^{\rho-\sigma}\mathfrak{C} \supseteqq \mathfrak{A}$. Hence $\mathfrak{p}^{\rho-\sigma}\mathfrak{C} \supseteqq \mathfrak{A}'$, and passing to the extended ideals in $\mathfrak{O}'$, we find $x'^{\rho}\mathfrak{B}_1 \supseteqq x'^{\rho}\mathfrak{B}^*$, i. e. $\mathfrak{B}_1 \supseteqq \mathfrak{B}^*$, q. e. d.

NOTE.—For those not familiar with the geometric terminology we give here the definition of the effective multiplicities on the basis of the present treatment. We associate with each simple v-ideal $\mathfrak{P}_{k+1}$ of kind $k+1$, belonging to the prime ideal $\mathfrak{P}_1 = \mathfrak{p} = (x, y)$, a point 0_{k+1} in the k-th neighborhood of the point $0_1(0, 0)$ of the (x, y)-plane. Let $\mathfrak{P}_1, \mathfrak{P}_2, \cdots, \mathfrak{P}_k$ be the simple v-ideals of kind $1, 2, \cdots, k$ determined by $\mathfrak{P}_{k+1}$ and preceding it (Theorem 6. 1), and let $0_1, 0_2, \cdots, 0_k$ be the associated points. We proceed to define the *set of base points* of the ideal $\mathfrak{P}_{k+1}$ or the symbol

$$B(\mathfrak{P}_{k+1}) = (0_1{}^{r_1}0_2{}^{r_2}\cdots 0_k{}^{r_k}0_{k+1}), \qquad r_i > 0,$$

and we shall say that r_i is the *effective multiplicity* of the ideal $\mathfrak{P}_{k+1}$ at $0_i (r_{k+1} = 1)$. We set $B(\mathfrak{P}_1) = (0_1)$. For any k we define $B(\mathfrak{P}_{k+1})$ by induction with respect to k. We know, by Theorem 6. 2, that if B is any valuation for which $\mathfrak{P}_{k+1}$ is a v-ideal, the v-ideals for B which precede $\mathfrak{P}_{k+1}$ are independent of B. Let $\mathfrak{q}_\lambda$ be the v-ideal which is followed immediately by $\mathfrak{P}_{k+1}$ and let $\mathfrak{q}_\lambda = \mathfrak{P}_1{}^{a_1}\mathfrak{P}_2{}^{a_2}\cdots\mathfrak{P}_k{}^{a_k}$. Assuming that the symbol $B(\mathfrak{P}_i)$ has already been defined for all $i < k+1$, we put

$$B(\mathfrak{P}_{k+1}) = [B(\mathfrak{P}_1)]^{a_1}[B(\mathfrak{P}_2)]^{a_2}\cdots[B(\mathfrak{P}_k)]^{a_k}O_{k+1},$$

where, if

$$B(\mathfrak{P}_{i+1}) = (0_1{}^{s_1}0_2{}^{s_2}\cdots 0_i{}^{s_i}0_{i+1})$$

and

$$B(\mathfrak{P}_{j+1}) = (0_1^{t_1} 0_2^{t_2} \cdots 0_j^{t_j} 0_{j+1}),$$

then

$$B(\mathfrak{P}_i) B(\mathfrak{P}_j) = (0_1^{s_1+t_1} 0_2^{s_2+t_2} \cdots).$$

To check this definition against the customary geometric definition, we point out two implications of our definition:

(1) *$\mathfrak{P}_{k+1}$ is exactly divisible by $\mathfrak{p}^{r_1}$*, i.e. $\mathfrak{P}_{k+1} \equiv 0(\mathfrak{p}^r)$, $\mathfrak{P}_{k+1} \not\equiv 0(\mathfrak{p}^{r+1})$. The assertion is true for $k = 0$. Assuming that is true for $\mathfrak{P}_{i+1}$, $i = 0, 1, \cdots, k-1$ and that $\mathfrak{P}_{i+1}$ is exactly divisible by $\mathfrak{p}^{r_{i1}}$, then

$$B(\mathfrak{P}_{i+1}) = (0_1^{r_{i1}} \cdots), \qquad r_1 = \alpha_1 r_{01} + \cdots + \alpha_k r_{k-1,1}$$

and from this we conclude, in view of $\mathfrak{q}_\lambda = \mathfrak{P}_1^{\alpha_1} \mathfrak{P}_2^{\alpha_2} \cdots \mathfrak{P}_k^{\alpha_k}$, that $\mathfrak{q}_\lambda$ is exactly divisible by $\mathfrak{p}^{r_1}$. Now $\mathfrak{p}\mathfrak{q}_\lambda \equiv 0(\mathfrak{P}_{k+1})$, since $\mathfrak{P}_{k+1}$ is a maximal subideal of $\mathfrak{q}_\lambda$. Hence $\mathfrak{P}_{k+1}$ is divisible at most by $\mathfrak{p}^{r_1+1}$. If $\mathfrak{P}_{k+1}$ was divisible by $\mathfrak{p}^{r_1+1}$, then the system of subforms $\Omega(\mathfrak{p}\mathfrak{q}_r)$ would have been a subsystem of $\Omega(\mathfrak{P}_{k+1})$. This is impossible, since $\Omega(\mathfrak{p}\mathfrak{q}_\lambda)$ is of dimension $\geqq 1$, while $\Omega(\mathfrak{P}_{k+1})$ is of dimension 0. Hence $\mathfrak{P}_{k+1}$ is divisible exactly by $\mathfrak{p}^{r_1}$. We have thus proved that in the general polynomial $f(x, y)$ in $\mathfrak{P}_{k+1}$ the terms of lowest degree are of degree r_1, i.e. *the curve $f = 0$ has at 0_r an r_1-fold point (while no curve $f = 0$, $f \varepsilon \mathfrak{P}_{k+1}$, has at 0_1 a multiplicity less than r_1).*

(2) *If $\mathfrak{P}'_i$ is the transform of $\mathfrak{P}_{i+1}$ by a quadratic transformation T having at 0_1 a fundamental point, then $B(\mathfrak{P}'_k) = (0'^{r_2}_1 \cdots 0'^{r_k}_{k-1} 0'_k)$, where $0'_i$ is the point associated with $\mathfrak{P}'_i$.* To prove this, we first observe that from the fact, just proved above, that $\mathfrak{q}_\lambda$ and $\mathfrak{P}_{k+1}$ are both divisible exactly by the same power, $\mathfrak{p}^{r_1}$, of $\mathfrak{p}$, it follows that $\Omega(\mathfrak{P}_{k+1})$ is a subsystem of $\Omega(\mathfrak{q}_\lambda)$. Since $\mathfrak{P}_{k+1}$ is a maximal subideal of $\mathfrak{q}_\lambda$, the dimension of $\Omega(\mathfrak{q}_\lambda)$ cannot exceed the dimension of $\Omega(\mathfrak{P}_{k+1})$ augmented by 1. But $\Omega(\mathfrak{P}_{k+1})$ is of dimension 0, while $\Omega(\mathfrak{q}_\lambda)$ is of dimension α_1. Hence $\alpha_1 \leqq 1$. Let $T(\mathfrak{q}_\lambda) = x'^{r_1} \mathfrak{q}'_\mu$, where $\mathfrak{q}'_\mu = \mathfrak{P}'^{\alpha_2}_1 \mathfrak{P}'^{\alpha_3}_2 \cdots \mathfrak{P}'^{\alpha_k}_{k-1}$. We assert that *$\mathfrak{q}'_\mu$ is the immediate predecessor of $\mathfrak{P}'_k$.* If $\alpha_1 = 0$, the assertion follows immediately from Theorem 4.4, since we have in this case $T^{-1}(\mathfrak{q}'_\mu) = \mathfrak{q}_\lambda$. Let now $\alpha_1 = 1$. Assume that $\mathfrak{q}'_\mu$ is not the immediate predecessor of $\mathfrak{P}'_k$ and let $\mathfrak{q}'_s$ be a v-ideal between $\mathfrak{q}'_\mu$ and $\mathfrak{P}'_k$. Let $\mathfrak{q}_m = T^{-1}(\mathfrak{q}'_\mu) = \mathfrak{P}_2^{\alpha_2} \cdots \mathfrak{P}_k^{\alpha_k}$, $\mathfrak{q}_r = T^{-1}(\mathfrak{q}'_s)$, whence $\mathfrak{q}_m \supset \mathfrak{q}_r \supset \mathfrak{P}_{k+1}$ (by Theorem 4.4). We have $\mathfrak{q}_r \equiv 0(\mathfrak{q}_m) \equiv 0(\mathfrak{p}^{r_1-1})$, and also $\mathfrak{q}_r \not\equiv 0(\mathfrak{p}^{r_1+1})$, since $\mathfrak{P}_{k+1} \equiv 0(\mathfrak{q}_r)$. We also have $\mathfrak{p}\mathfrak{q}_m = \mathfrak{q}_\lambda \equiv 0(\mathfrak{q}_r)$, since $\mathfrak{q}_\lambda$ is the immediate predecessor of $\mathfrak{P}_{k+1}$. If $\mathfrak{q}_r$ was divisible by $\mathfrak{p}^{r_1}$, then $\Omega(\mathfrak{q}_\lambda)$ would be a subsystem of $\Omega(\mathfrak{q}_r)$, and this is impossible, since $\Omega(\mathfrak{q}_\lambda)$ is of dimension 1, while $\Omega(\mathfrak{q}_r)$ must be of dimension 0. Hence $\mathfrak{q}_r$ is divisible exactly by $\mathfrak{p}^{r_1-1}$. Now we have $v(\mathfrak{q}_r) > v(\mathfrak{q}_m)$, $v(\mathfrak{p}\mathfrak{q}_r) > v(\mathfrak{p}\mathfrak{q}_m) = v(\mathfrak{q}_\lambda)$, and consequently $\mathfrak{p}\mathfrak{q}_r \equiv 0(\mathfrak{P}_{k+1})$. This is a contradiction, since both $\mathfrak{p}\mathfrak{q}_r$ and $\mathfrak{P}_{k+1}$ are divisible exactly by $\mathfrak{p}^{r_1}$ and $\Omega(\mathfrak{p}\mathfrak{q}_r)$ is of dimension 1. It is thus proved that $\mathfrak{q}'_\mu$ is the immediate predecessor of $\mathfrak{P}'_k$. As a consequence we have

$$B(\mathfrak{P}'_k) = [B(\mathfrak{P}'_1)]^{\alpha_2} [B(\mathfrak{P}'_2)]^{\alpha_3} \cdots [B(\mathfrak{P}'_{k-1})]^{\alpha_k} 0'_k.$$

From this relation our statement follows immediately by induction with respect to k. Our result implies that *if the general curve $f = 0$, $f \varepsilon \mathfrak{P}_{k+1}$, passes through $0_1, 0_2, \cdots, 0_{k+1}$ with effective multiplicities $r_1, r_2, \cdots, r_k, 1$, then its transform by*

the quadratic transformation T passes through the points $0'_1, 0'_2, \cdots, 0'_k$ with effective multiplicities $r_2, r_3, \cdots, r_k, 1$. The identity between our definition of effective multiplicities and the customary geometric definition is thus fully proved. It is hardly necessary to add that the definition of the symbol $B(\mathfrak{A})$ for any complete ideal amounts to postulating the relation $B(\mathfrak{B}\mathfrak{C}) = B(\mathfrak{B})B(\mathfrak{C})$.

13. Simple v-ideals and divisors of the second kind. Let $\mathfrak{P}$ be a divisor of the field $\Sigma = \mathfrak{k}(x, y)$, i. e. an homomorphism of Σ upon a field Σ' of dimension 1 (and the symbol ∞), and let us assume that $\mathfrak{P}$ is of second kind for the ring $\mathfrak{O} = \mathfrak{k}[x, y]$, i. e. that the prime ideal $\mathfrak{p}$ determined in $\mathfrak{O}$ by the divisor $\mathfrak{P}$ is zero-dimensional, say $\mathfrak{p} = (x, y)$ (we exclude the case in which x or y are mapped upon ∞). The points of the Riemann surface of the field Σ' define a set of valuations $\{B_\alpha\}$ of the field Σ, all of rank 2. Let $\{\mathfrak{q}_i^{(\alpha)}\}$ be the Jordan sequence of v-ideals in $\mathfrak{O}$ belonging to the valuation B_α. The ideal $\mathfrak{q}_1^{(\alpha)} = \mathfrak{p}$ is independent of α. There may be other values of i such that $\mathfrak{q}_i^{(\alpha)}$ is independent of α, and, in particular, there may occur in the sequence $\{\mathfrak{q}_i^{(\alpha)}\}$ simple ideals independent of α. Their number is necessarily finite, because the simple v-ideals in the sequence $\{\mathfrak{q}_i^{(\alpha)}\}$ determine the sequence completely (Theorem 6. 2) and since the valuations B_α are all distinct. Let $\mathfrak{P}_\rho$ be the last simple v-ideal, of kind ρ, which occurs in all the sequences $\{\mathfrak{q}_i^{(\alpha)}\}$. The simple ideal $\mathfrak{P}^{(\alpha)}_{\rho+1}$, of kind $\rho + 1$, will then vary with α. We have thus associated with every divisor of Σ, of the second kind, with respect to $\mathfrak{O}$, a simple v-ideal $\mathfrak{P}_\rho$ in $\mathfrak{O}$. If we apply ρ successive quadratic transformations, getting a polynomial ring $\mathfrak{k}[X, Y]$ of the new indeterminates X, Y, the ideal $\mathfrak{P}^{(\alpha)}_{\rho+1}$ is transformed into a prime 0-dimensional ideal $\mathfrak{p}^{(\alpha)} = (X, Y - c^{(\alpha)})$, and the constant $c^{(\alpha)}$ must vary as α varies. As a consequence, the divisor $\mathfrak{P}$ is of the first kind with respect to the ring $\mathfrak{k}[X, Y]$, and the corresponding prime ideal in this ring is necessarily the 1-dimensional ideal (X). This shows that there exists a divisor $\mathfrak{P}$ for any preassigned simple v-ideal $\mathfrak{P}_\rho$ and that $\mathfrak{P}$ is uniquely determined by $\mathfrak{P}_\rho$. *We have then a one to one correspondence between the divisors of the second kind (with respect to the ring $\mathfrak{k}[x, y]$) and the 0-dimensional simple v-ideals in $\mathfrak{k}[x, y]$.* The field Σ' upon which Σ is mapped by $\mathfrak{P}$ coincides with the field $\mathfrak{k}(Y)$ and is therefore purely transcendental.

It is important to point out that as the valuation B_α runs through the set $\{B_\alpha\}$ determined by the points of the field Σ', the set $\{\mathfrak{P}^{(\alpha)}_{\rho+1}\}$ will include *all* the simple v-ideals $\mathfrak{P}_{\rho+1}$ of kind $\rho + 1$, such that $\mathfrak{P}_{\rho+1}$ and $\mathfrak{P}_\rho$ belong together to one and the same valuation. This follows from the fact, that all such ideals $\mathfrak{P}_{\rho+1}$ are transformed by ρ successive quadratic transformations into the above ideals $\mathfrak{p}^{(\alpha)}$.

The preceding considerations refer to the field $\mathfrak{k}(x, y)$ of rational func-

tions in x, y, but can be immediately extended to the field Σ^* of meromorphic functions in x, y. If we put $x = x$, $y = x'y'$, every holomorphic function $\xi(x, y)$ assumes the form $x'^{\rho}(f_1(y') + x'f_2(y') + \cdots)$. If η is another holomorphic function of x, y, and $\eta = x'^{\sigma}(\phi_1(y') + x'\phi_2(y') + \cdots)$, we map the function ξ/η upon 0, ∞ or $f_1(y')/\phi_1(y')$, according as $\rho > \sigma$, $\rho < \sigma$ or $\rho = \sigma$. This mapping defines an homomorphism of the field Σ^* upon the purely transcendental field $\mathfrak{k}(y')$. We regard this homomorphism as a divisor of the second kind of Σ^*, since the prime ideal in $\mathfrak{k}\{x, y\}$ determined by this divisor is the 0-dimensional ideal $\mathfrak{p}^* = (x, y)$. We associate this divisor with the simple ideal $\mathfrak{p}^*$. In the same manner we may associate with any simple ideal $\mathfrak{P}_{h+1}$ of kind $h + 1$ in $\mathfrak{k}\{x, y\}$ a divisor of Σ^*, in which Σ^* is mapped upon a purely transcendental field of one variable. We do this by first applying h successive quadratic transformations, getting a ring $\mathfrak{O}^*_h = \mathfrak{k}\{x_h, y_h\}$, which contains $\mathfrak{k}\{x, y\}$ and in which the ideal $\mathfrak{P}_{h+1}$ corresponds to a simple v-ideal of kind 1, i. e. to the ideal (x_h, y_h). This ideal defines a divisor of the field Σ^*_h of meromorphic functions of x_h, y_h, mapping Σ^*_h upon the field $\mathfrak{k}(y'_h)$ of rational functions of y'_h $(= y_h/x_h)$. In this homomorphism the subfield Σ^* of Σ^*_h is mapped upon the entire field $\mathfrak{k}(y'_h)$, since x_h, y_h are rational functions of x and y. We associate the divisor of Σ^*, obtained in this manner, with the simple v-ideal $\mathfrak{P}_{h+1}$. In exactly the same manner as for the field of rational functions of x and y, it is shown that the correspondence between all the divisors of Σ^*, defined by homomorphic mappings of Σ upon fields of dimension one with respect to $\mathfrak{k}$ and of second kind with respect to $\mathfrak{k}\{x, y\}$, and the simple v-ideals in $\mathfrak{k}\{x, y\}$ is $(1, 1)$, and that consequently any such divisor is purely transcendental (i. e. the field upon which Σ^* is mapped by a divisor of the second kind is necessarily purely transcendental).

We point out explicitly, that if a divisor of Σ^* *of the first kind with respect to* $\mathfrak{k}\{x, y\}$ is defined as an homomorphic mapping of Σ^* upon a field Ω, such that the prime ideal determined in $\mathfrak{k}\{x, y\}$ by the divisor is one-dimensional, then Ω is necessarily the field of all meromorphic functions of one indeterminate. Thus for fields of meromorphic functions Σ^* the classification of the divisors into two kinds is not merely a relative classification with respect to the ring $\mathfrak{k}\{x, y\}$, but rather a classification in terms of the properties of the field Σ^* itself. That this should be so is only natural, in view of the privileged rôle which the ring of holomorphic functions plays in the field Σ^*.

We conclude with one final remark. which finds an application in the proof of the well-known algebro-geometric theorem, that a pencil of curves on an algebraic surface is necessarily linear, if the pencil has a base point at a simple point of the surface. This remark will be elaborated in a joint note

by Dr. O. Schilling and the present author. At this place we wish only to observe that the proof of this theorem is based upon the following assertion:

Given a meromorphic function $f(x,y)/\phi(x,y)$, and assuming that the elements f and ϕ of $\mathfrak{k}\{x,y\}$ are relatively prime and that neither is a unit, then there exists a divisor $\mathfrak{P}$ of second kind of the field Σ^* of meromorphic functions, such that f/ϕ is mapped upon a transcendental element of the image field. This assertion can be readily proved as follows. Consider the ideal $\mathfrak{A}=(f,\phi)$ in $\mathfrak{k}\{x,y\}$. This ideal is zero-dimensional, since f and ϕ are relatively prime and are both contained in the ideal (x,y). Let $\mathfrak{A}'$ be the complete ideal determined by $\mathfrak{A}$. Let us first consider the case in which $\mathfrak{A}'=\mathfrak{p}^*=(x,y)$. In this case we assert that the required divisor $\mathfrak{P}$ is the one associated with $\mathfrak{p}^*$. In fact, assume that f/ϕ is mapped by $\mathfrak{P}$ upon a constant c. We may assume $c=0$, replacing $f-c\phi$ by f. Under this hypothesis f/ϕ will have positive value in all the zero-dimensional valuations B_a defined by the points of the Riemann surface of $\mathfrak{P}$, and hence, since $\phi\equiv 0(\mathfrak{p}^*)$, f must belong, for any α, to the valuation ideal $\mathfrak{q}_2^{(\alpha)}$ which follows $\mathfrak{p}^*$ in the Jordan sequence of v-ideals belonging to B_α. As α varies, $\mathfrak{q}_2^{(\alpha)}$ can be any maximal subideal of $\mathfrak{p}^*$ (see section 8) and is a simple ideal (of kind 2). Now for *some* α we will have $\phi\equiv 0(\mathfrak{q}_2^{\alpha})$. In fact, it is sufficient to consider the valuation defined by an irreducible branch of the analytical locus $\phi=0$. For this valuation it is true that the element ϕ is contained in *all* the 0-dimensional v-ideals $\mathfrak{q}_i$ belonging to the valuation, whence also in $\mathfrak{q}_2$. It follows then that for some α, both f and ϕ are contained in $\mathfrak{q}_2^{(\alpha)}$. But this is impossible, since $\mathfrak{q}_2^{(\alpha)}$ is a complete ideal which does not contain the complete ideal $\mathfrak{A}'=\mathfrak{p}^*$ determined by (f,ϕ).

In the general case, let $\mathfrak{A}'=\mathfrak{P}_1^{a_1}\mathfrak{P}_2^{a_2}\cdots\mathfrak{P}_k^{a_k}$, where $\mathfrak{P}_i$ is a simple v-ideal of kind h_i. Let $h=\max\{h_i\}$, and let us apply the quadratic transformation $x'=x$, $y'=y/x$. If $\mathfrak{A}\equiv 0(\mathfrak{p}^{*\rho})$, $\mathfrak{A}\not\equiv 0(\mathfrak{p}^{*\rho+1})$, then by Theorem 12.3, also $\mathfrak{A}'\equiv 0(\mathfrak{p}^{*\rho})$, $\mathfrak{A}'\not\equiv 0(\mathfrak{p}^{*\rho+1})$, and putting $f=x'^\rho f_1$, $\phi=x'^\rho\phi_1$, $\mathfrak{A}_1=(f_1,\phi_1)$, the complete ideal $\mathfrak{A}'_1$ determined by $\mathfrak{A}_1$ is the transform $T(\mathfrak{A}')$, i. e. the ideal

$$\mathfrak{A}'_1=\mathfrak{P}'^{a_1}_1\mathfrak{P}'^{a_2}_2\cdots\mathfrak{P}'^{a_k}_k.$$

Here each $\mathfrak{P}'_i$ is of kind h_i-1, so that $\max\{h_i-1\}=h-1$. Since the theorem has already been proved for $h=1$ ($\mathfrak{A}'=\mathfrak{p}^*$), we may assume that the theorem is true for $h-1$. We may even assume that for the function f_1/ϕ_1 the divisor whose existence is stated in our assertion is the divisor associated with a factor $\mathfrak{P}'_s$, such that $\mathfrak{P}'_s$ is of maximum kind $h-1$. Then it follows immediately that the divisor of the field Σ^* associated with the factor $\mathfrak{P}_s$ satisfies the assertion.

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INTEGRAL FORMS AND VARIATIONAL ORTHOGONALITY.*

By PHILIP HARTMAN and RICHARD KERSHNER.

Introduction. The arc-length of a curve $x_1 = x_1(t)$; $x_2 = x_2(t)$, if it exists, is defined [1] by an integral of the form

$$L(x_1, x_2) = \int_a^b \{(dx_1)^2 + (dx_2)^2\}^{\frac{1}{2}}, \tag{1}$$

where (1) means the limit of approximating sums

$$\Sigma\{\Delta x_1)^2 + (\Delta x_2)^2\}^{\frac{1}{2}}$$

of the Riemann-Stieltjes type. It is well known that (1) exists whenever x_1 and x_2 are of bounded variation in $[a, b]$; and that, in case x_1 and x_2 are absolutely continuous, (1) reduces to the ordinary Lebesgue integral

$$L(x_1, x_2) = \int_a^b \{(x'_1(t))^2 + (x'_2(t))^2\}^{\frac{1}{2}} dt. \tag{2}$$

In this paper the integral (1) will be investigated in cases when x_1 and x_2 are not absolutely continuous but possess, under the Lebesgue decomposition, a purely singular or a purely discontinuous component. The results to be obtained furnish means for the calculation of the length of curves given by a parametric representation for which (2) is not valid. These results will be based on the fact to be proved that, while (1) is clearly not decomposed linearly under arbitrary linear decompositions of x_1, x_2, nevertheless, for the Lebesgue decomposition, $x_i(t) = a_i(t) + s_i(t) + p_i(t)$, where a_i is absolutely continuous, s_i is purely singular,[2] and p_i is purely discontinuous, one has

$$L(x_1, x_2) = L(a_1, a_2) + L(s_1, s_2) + L(p_1, p_2). \tag{3}$$

It should be noticed that, since $L(a_1, a_2)$ reduces to an ordinary Lebesgue integral and $L(p_1, p_2)$ degenerates into an infinite sum, the usefulness of (3) is limited largely by the difficulty of calculating $L(s_1, s_2)$. Of course, in special cases it may be possible to eliminate the parameter between $s_1(t)$ and $s_2(t)$ and reduce this integral to one of the Lebesgue type. Additional results which sometimes lead to the explicit evaluation of $L(s_1, s_2)$ will be given in

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[1] Cf., e. g., Saks [7], p. 57.

[2] A purely singular function will be supposed to be continuous.

the sequel. In this direction the following fact may be mentioned here: If, for every t in $[a, b]$, either $s'_1(t) = 0$ or $s'_2(t) = 0$, then

$$L(s_1, s_2) = V\{[a, b]; s_1\} + V\{[a, b]; s_2\},$$

where $V\{[a, b]; f\}$ is the total variation of f on $[a, b]$.

In the non-parametric case, the length $L(y)$ of the continuous curve $y = y(x)$ is given by

$$L(y) = \int_a^b \{1 + (y'(t))^2\}^{\frac{1}{2}} dt + V\{[a, b]; y\} - \int_a^b | y'(t) | dt.$$

In particular, if $y(x)$ is purely singular and monotone in $[a, b]$,

$$L(y) = | y(b) - y(a) | + | b - a |.$$

This last statement is trivial in the case that $y(x)$ is almost everywhere constant.

At the suggestion of Professor Wintner the proofs of the facts mentioned above have been extended so as to apply, not only to the Euclidean arc-length (1), but to very general integral forms of the type

$$(4) \qquad \Phi(\mathbf{X}) = \Phi\{[a, b); \mathbf{X}\} = \int_a^b F(\mathbf{Y}; d\mathbf{X}),$$

where

$$\mathbf{Y} = \mathbf{Y}(t) = \{y_1(t), \cdots, y_m(t)\} \quad \text{and} \quad \mathbf{X} = \mathbf{X}(t) = \{x_1(t), \cdots, x_n(t)\}$$

are vector functions of t on $[a, b]$ and where

$$F(\mathbf{Y}; k\mathbf{Z}) = kF(\mathbf{Y}; \mathbf{Z}), \qquad (k \geqq 0).$$

The best known integrals of this type are, of course, the Stieltjes integral

$$(5) \qquad \Phi(\mathbf{X}) = \int_a^b y(t) dx(t)$$

and the total variation of the function $x(t)$

$$(6) \qquad \Phi(\mathbf{X}) = V\{[a, b); x\} = \int_a^b | dx |.$$

Another simple example is the case

$$(7) \qquad \Phi(\mathbf{X}) = \int_a^b \{\sum_{j=1}^n | dx_j |^p\}^{1/p}, \qquad (p > 1),$$

of which the case $p = 2$ is Euclidean arc-length. The Riemannian arc-length

$$(8) \qquad \Phi(\mathbf{X}) = \int_a^b \{\sum_{i,k=1}^n g_{ik} dx_i dx_k\}^{\frac{1}{2}}$$

and the Finsler metrics

$$\Phi(X) = \int_a^b F(X; dX) \tag{9}$$

can also be considered as special cases of (4).

In connection with Hilbert's theory of bounded quadratic forms with continuous spectra, Hellinger [4] and Hahn [2] consider integrals of the form

$$\Phi(X) = H_1(x_1, x_2) = \int_a^b (dx_1)^2/dx_2 \tag{10}$$

and

$$\Phi(X) = H_2(x_1, x_2) = \int_a^b | dx_1 dx_2 |^{\frac{1}{2}}, \tag{11}$$

and need, in order to describe the system of orthogonal invariants, especially a study of the case that $X(t)$ is not absolutely continuous. The general results to be obtained provide an essential simplification of the use of these integrals in this connection.

In general, it is found, in conformity with (3), that if

$$X(t) = X_1(t) + X_2(t) + X_3(t), \tag{12}$$

where $X(t)$ is of bounded variation, $X_1(t)$ is absolutely continuous, $X_2(t)$ is purely singular, and $X_3(t)$ is purely discontinuous, then, under very general conditions on the function F occurring in (4),

$$\Phi(X) = \Phi(X_1) + \Phi(X_2) + \Phi(X_3). \tag{13}$$

In particular (13) holds when $\Phi(X)$ is any one of the explicit types (5), (6), (7), (8), (10), (11). In the case of (9) some restrictions on F are, of course, required.

Actually the Lebesgue decomposition (12) is only a special case of a large class of decompositions for which the linear property exhibited in (13) is demonstrated. Other decompositions, of a similar nature, are considered for which (13) does not hold precisely but requires corrective terms.

Again in conformity with the case of Euclidean arc-length, it is shown that the first integral $\Phi(X_1)$ occurring on the right side of (13) can be reduced, in all cases for which (13) is proved, to an ordinary Lebesgue integral with respect to t, namely

$$\Phi(X_1) = \int_a^b F(Y(t);\ X'(t))dt, \qquad (X' = dX/dt).$$

Furthermore, the last integral, $\Phi(X_3)$, occurring on the right of (13) degenerates to an ordinary series of the form

$$\Phi(X_3) = \Sigma F(Y(\rho_i); X(\rho_i + 0) - X(\rho_i - 0)),$$

where the sum is taken over all discontinuity points ρ_i of $X(t)$. Thus (13), in a sense, reduces the investigation of integrals of type (4) to the case that $X(t)$ is purely singular, i. e., that all components of $X(t)$ are purely singular.

The results to be obtained are, of course, trivial in the Stieltjes case (5) and essentially known in the case (6) of total variation but in all other cases they seem to provide a new analysis of the integrals under consideration.

1. Definitions and conditions. The integral (4) may be defined as a direct generalization of the Riemann-Stieltjes integral in the following manner: Let $a = t_0 < t_1 < \cdots < t_{p+1} = b$ be a mesh on the interval $[a, b]$. Then $\Phi(X)$ is the limit of an arbitrary sequence of sums

$$(14) \quad \sum_{j=0}^{p} F(Y(\xi_j); \Delta_j X), \quad (\Delta_j X = X(t_{j+1}) - X(t_j); \ t_j \leqq \xi_j < t_{j+1}),$$

associated with any sequence of meshes for which $\Delta = \max | t_{j+1} - t_j | \to 0$, provided this limit exists and is independent of the particular choice of the sequence of meshes and of the intermediary values ξ_j. Correspondingly, the integral (4) is said to exist if the sums (14) have such a unique limit. The integral (4) will be said to exist absolutely if

$$(15) \quad \int_a^b | F(Y; dX) |$$

as well as (4), exists. The corresponding generalization of the Lebesgue-Stieltjes type will not be considered here.

It will be convenient to list certain of the conditions which will be used in the sequel.

Throughout the paper it will be supposed that $X(t)$ is of bounded variation, i. e., each component $x_j(t)$ is of bounded variation, in $[a, b]$. It will also be supposed that $X(t)$ is continuous from the left. This involves no loss of generality, since it necessitates, at most, the redefinition of $X(t)$ at an enumerable set of points, a process which does not affect the value of the integral (4).

No restrictions will be imposed on $Y(t)$ other than that its components be Baire functions. Even this requirement is unnecessary for many purposes.

Of $F(Y; Z)$, in addition to

(A) Positive linear homogeneity: $F(Y; kZ) = kF(Y; Z)$, $(k \geqq 0)$, which has already been mentioned, it will always be supposed that F is continuous in the n variables z_j together.

Conditions which will be required not always but on occasion, are the following:

(B) Polar symmetry: $F(\mathbf{Y}(t)\,;\,-\mathbf{Z}) = F(\mathbf{Y}(t)\,;\,\mathbf{Z})$.

(C) Convexity: $F(\mathbf{Y}(t)\,;\,\mathbf{Z}_1 + \mathbf{Z}_2) \leqq F(\mathbf{Y}(t)\,;\,\mathbf{Z}_1) + F(\mathbf{Y}(t)\,;\,\mathbf{Z}_2)$.
In virtue of (A), for $k = \frac{1}{2}$, this condition is equivalent to the usual notion of convexity with respect to $\mathbf{Z}$.

(D) For every $\epsilon > 0$, one can choose a $\delta_\epsilon > 0$ such that

$$\Big|\sum_{j=1}^{\infty} F(\mathbf{Y}(\xi_j)\,;\,\mathbf{Z}_j)\Big| < \epsilon \quad \text{if} \quad \sum_{j=1}^{\infty}\sum_{k=1}^{n} |z_{jk}| < \delta_\epsilon,$$

where $\mathbf{Z}_j = \{z_{j1}, \cdots, z_{jn}\}$ and $a \leqq \xi_j \leqq b$.

(D′) If $\mathbf{Y}(t)$ and $\mathbf{X}(t)$ are vector functions for which (4) exists, then for every $\epsilon > 0$, there exists a $\delta_\epsilon > 0$ such that

$$\Big|\sum_{j=1}^{\infty} F(\mathbf{Y}(\xi_j)\,;\,\Delta_j\mathbf{X})\Big| < \epsilon \quad \text{if} \quad \sum_{j=1}^{\infty}\sum_{k=1}^{n} |\Delta_j x_k| < \delta_\epsilon,$$

where $\{[t_j, t'_j)\}$ is any set of non-overlapping, half-open intervals on $[a, b]$, $\Delta_j x_k = x_k(t'_j) - x_k(t_j)$, and $t_j \leqq \xi_j < t'_j$.

(E) For every $\epsilon > 0$, there exists a $\delta_\epsilon > 0$ such that

$$\left|\sum_{j=1}^{\infty}\{F(\mathbf{Y}(\xi_j)\,;\,\mathbf{Z}_j) - F(\mathbf{Y}(\xi_j)\,;\,\mathbf{W}_j)\}\right| < \epsilon \quad \text{if} \quad \sum_{j=1}^{\infty}\sum_{k=1}^{n} |z_{jk} - w_{jk}| < \delta_\epsilon,$$

where $\mathbf{Z}_j = \{z_{j1}, \cdots, z_{jn}\}$, $\mathbf{W}_j = \{w_{j1}, \cdots, w_{jn}\}$, and $a \leqq \xi_j \leqq b$.

(E′) If $\mathbf{Y}(t), \mathbf{X}_1(t), \mathbf{X}_2(t)$ are vector functions for which $\int_a^b F(\mathbf{Y}\,;\,d\mathbf{X}_1)$ and $\int_a^b F(\mathbf{Y}\,;\,d\mathbf{X}_2)$ exist, then one can choose, for every $\epsilon > 0$, a $\delta_\epsilon > 0$ such that

$$\left|\sum_{j=1}^{\infty}\{F(\mathbf{Y}(\xi_j)\,;\,\Delta_j\mathbf{X}_1) - F(\mathbf{Y}(\xi_j)\,;\,\Delta_j\mathbf{X}_2)\}\right| < \epsilon$$
$$\text{if} \quad \sum_{j=1}^{\infty}\sum_{k=1}^{n} |\Delta_j x_{1k} - \Delta_j x_{2k}| < \delta_\epsilon,$$

the notations being analogous to those used in (D′).

The relations between these last four conditions are indicated by the following table of implications,

$$(\mathrm{E}) \cdot\supset\cdot (\mathrm{E}'),\ (\mathrm{D}) \cdot\supset\cdot (\mathrm{D}'),\ (\mathrm{E}) \cdot\supset\cdot (\mathrm{D}),\ (\mathrm{E}') \cdot\supset\cdot (\mathrm{D}'),$$

where, e.g., $(\mathrm{E}) \cdot\supset\cdot (\mathrm{E}')$ means that (E′) is satisfied whenever (E) is. Condition (D) is somewhat weaker than a Lipschitz condition on $F(\mathbf{Y}(t)\,;\,\mathbf{Z})$

with respect to the n variables z_j at the point $z_j = 0$, which is uniform with respect to the parameter t in the t-interval $[a, b]$. Correspondingly, (E) is satisfied if $F(\mathbf{Y}(t); \mathbf{Z})$ fulfills a uniform Lipschitz condition in the n independent variables z_j.

2. $\Phi(\mathbf{X})$ as a set function. For later purposes it will be convenient to extend the definition of the integral $\Phi(\mathbf{X}) = \Phi\{[a, b); \mathbf{X}\}$ to a set function $\Phi\{S; \mathbf{X}\}$, defined for all Borel sets S in $[a, b]$. It is known[3] that if the point function $f(u; \mathbf{X}) = \Phi\{[a, u); \mathbf{X}\}$ exists for every u in $a < u \leqq b$, is of bounded variation, and continuous from the left in $a < u \leqq b$, then there exists a unique, bounded, completely additive set function $\Phi\{S; \mathbf{X}\}$, defined for all Borel sets S in $[a, b]$, such that if S is the half-open interval $[a, u)$, $a \leqq t < u$, then $\Phi\{S; \mathbf{X}\} = f(u; \mathbf{X})$.

This set function $\Phi\{S; \mathbf{X}\}$ may be obtained in the following way: Let

$$f(u; \mathbf{X}) = f_1(u; \mathbf{X}) - f_2(u; \mathbf{X}),$$

where f_1 and f_2 are non-decreasing functions of u in $[a, b]$ and let

$$\Phi_i\{S; \mathbf{X}\} = \text{g. l. b.} \sum_k [f_i(u'_k) - f_i(u_k)], \qquad (i = 1, 2),$$

where $a \leqq u_k < u'_k \leqq b$ and $S \subset \sum_k [u_k, u'_k)$. Then

$$\Phi\{S; \mathbf{X}\} = \Phi_1\{S; \mathbf{X}\} - \Phi_2\{S; \mathbf{X}\}.$$

When such an extension is possible, the notation

$$\Phi\{S; \mathbf{X}\} = \int_S F(\mathbf{Y}; d\mathbf{X}) \tag{16}$$

will be used, in conformity with (4).

Now there will be proved

THEOREM I. *If* (4) *exists absolutely and F satisfies condition* (D'), *then*

$$f(u; \mathbf{X}) = \int_a^u F(\mathbf{Y}; d\mathbf{X}) \tag{17}$$

exists for every u in $a < u \leqq b$ and there exists a completely additive set function (16) *defined for all Borel sets in* $[a, b]$ *such that*

$$\Phi\{[a, u); \mathbf{X}\} = \int_a^u F(\mathbf{Y}; d\mathbf{X}).$$

Proof. The fact that (17) exists for every u in $[a, b]$ is, of course, a

[3] Cf., e. g., Radon [6], pp. 1305-1313.

consequence of the existence of $f(b;X)$. In fact, for every $\epsilon > 0$, one can choose a $\Delta_\epsilon > 0$ such that if

$$a = t_0 < t_1 < \cdots < t_{p+1} = b \tag{18}$$

is any mesh for which the degree of fineness Δ satisfies

$$\Delta = \max | t_{j+1} - t_j | < \Delta_\epsilon, \tag{19}$$

then

$$\Big| \sum_{j=0}^{p} F(Y(\xi_j);\, \Delta_j X) - \int_a^b F(Y; dX) \Big| < \epsilon \tag{20}$$

for arbitrary intermediary values ξ_j. Now if

$$a = t_0 < t_1 < \cdots < t_{k+1} = u < t_{k+2} < \cdots < t_{p+1} = b$$

and

$$a = t'_0 < t'_1 < \cdots < t'_{r+1} = u < t'_{r+2} = t_{k+2} < t'_{r+3} = t_{k+3} < \cdots < t'_{s+1} = b$$

are any two meshes having degrees of fineness Δ, Δ' respectively, which satisfy (19), then

$$\Big| \sum_{j=0}^{k} F(Y(\xi_j);\, \Delta_j X) + \sum_{j=k+1}^{p} F(Y(\xi_j);\, \Delta_j X) - \int_a^b F(Y; dX) \Big| < \epsilon,$$

$$\Big| \sum_{j=0}^{r} F(Y(\xi'_j);\, \Delta'_j X) + \sum_{j=k+1}^{p} F(Y(\xi_j);\, \Delta_j X) - \int_a^b F(Y; dX) \Big| < \epsilon.$$

Consequently,

$$\Big| \sum_{j=0}^{k} F(Y(\xi_j);\, \Delta_j X) - \sum_{j=0}^{r} F(Y(\xi'_j);\, \Delta'_j X) \Big| < 2\epsilon \tag{21}$$

whenever $\Delta < \Delta_\epsilon$ and $\Delta' < \Delta_\epsilon$. Thus (17) exists for every u in $[a, b]$.

As was mentioned at the beginning of this section, in order to complete the proof of Theorem I, it is sufficient to show that the function (17) is of bounded variation and continuous from the left on $a < u \leqq b$. That $f(u;X)$ is of bounded variation may be seen from the decomposition

$$f(u;X) = \int_a^u | F(Y; dX) | - \int_a^u \{ | F(Y; dX) | - F(Y; dX) \}$$

of f into the difference of two monotone functions. The existence of the integrals on the right side is assured by the assumption of the existence of (4) and (15), as has been shown above.

Since $X(t)$ is of bounded variation and continuous from the left, for every u, $a < u \leqq b$, there exists an $\eta = \eta_{\epsilon,u}$ such that the variation

$$V\{[u', u);\, X\} = \int_{u'}^{u} \sum_{k=1}^{n} | dx_k | < \delta_\epsilon \text{ if } 0 < u - u' < \eta,$$

where δ_ϵ is defined in (D′). In virtue of (D′), it is seen for any mesh $u' = t_0 < t_1 < \cdots < t_{p+1} = u$, that

$$\left| \sum_{j=0}^{p} F(\mathbf{Y}(\xi_j)\,;\,\Delta_j \mathbf{X}) \right| < \epsilon, \text{ since } \sum_{j=0}^{p} \sum_{k=1}^{n} | \Delta_j x_k | \leqq V\{[u', u)\,;\,X\} < \delta_\epsilon.$$

This implies that

$$| f(u;\mathbf{X}) - f(u';\mathbf{X}) | = \left| \int_{u'}^{u} F(\mathbf{Y};d\mathbf{X}) \right| < \epsilon \text{ if } 0 < u - u' < \eta,$$

completing the proof of Theorem I.

Obvious modifications of the proof of the last part of Theorem I show that if S is any Borel set for which $V\{S;\mathbf{X}\} = 0$, then $\Phi\{S;\mathbf{X}\} = 0$. In particular one has

THEOREM II. *Under the assumptions of Theorem I, the functions $f(u;\mathbf{X})$, $\Phi\{S;\mathbf{X}\}$ are absolutely continuous, purely singular, or purely discontinuous with $\mathbf{X}(t)$.*

Similarly, the first part of Theorem I can easily be extended to show that if S is a set consisting of a finite number of disjoint half-open intervals $[u_j, u'_j)$, $j = 1, \cdots, s$, and if

$$u_j = t_{j0} < t_{j1} < \cdots < t_{jp_{j+1}} = u'_j$$

is a mesh over S such that the degree of fineness Δ satisfies $\Delta < \Delta_\epsilon$, then

$$\left| \sum_{j=1}^{s} \sum_{k=0}^{p_j} F(\mathbf{Y}(\xi_{j,k})\,;\,\mathbf{X}(t_{j,k+1}) - \mathbf{X}(t_{j,k})) - \int_S F(\mathbf{Y};d\mathbf{X}) \right| < 2\epsilon.$$

Use of this fact will be made in the proof of Theorem III.

3. Reduction to other forms. It has been shown that in case $\mathbf{X}(t)$ is absolutely continuous, the integral (4) reduces in many cases to an ordinary Lebesgue integral. The precise formulation of the result to be proved in this direction is

THEOREM III. *If the assumptions of Theorem I are satisfied and if $\mathbf{X}(t)$ is absolutely continuous, then*

$$\int_a^b F(\mathbf{Y};d\mathbf{X}) = \int_a^b F(\mathbf{Y}(t)\,;\,\mathbf{X}'(t))dt. \tag{22}$$

For the proof of this theorem, the following lemma will be needed.

LEMMA 1. *If the assumptions of Theorem III are satisfied, then for any Borel set T on which $\mathbf{X}'(t)$ exists and*

$$(23) \qquad \lambda \leqq F(\mathbf{Y}(t)\,;\ \mathbf{X}'(t)) < \mu$$

one has

$$(24) \qquad \lambda \operatorname{meas} T \leqq \Phi\{T\,; \mathbf{X}\} \leqq \mu \operatorname{meas} T.$$

Proof. By Theorem II, the function $\Phi\{S; \mathbf{X}\}$ is absolutely continuous, i. e., for every $\epsilon > 0$, one can choose an $\eta_\epsilon > 0$ such that $|\,\Phi\{S; \mathbf{X}\}\,| < \epsilon$ when S is any Borel set for which meas $S < \eta_\epsilon$.

Now, since the set T occurring in the statement of the lemma is a bounded set, the covering theorem of Vitali implies that there exists a set $U = \sum_{j=1}^{p} [u_j, u'_j)$ such that

$$(25) \qquad 0 < u'_j - u_j < \Delta_\epsilon; \qquad u'_j \leqq u_{j+1};$$

$$(26) \qquad \operatorname{meas}(T - TU) < \eta_\epsilon; \qquad \operatorname{meas}(U - TU) < \eta_\epsilon;$$

$$(27) \qquad |\operatorname{meas} T - \operatorname{meas} U\,| < \epsilon;$$

$$(28) \qquad \lambda - \epsilon \leqq F(\mathbf{Y}(\xi_j)\,;\ [\mathbf{X}(u'_j) - \mathbf{X}(u_j)]/[u'_j - u_j]) < \mu; \qquad u_j \leqq \xi_j < u'_j.$$

Now, by (28) and the property (A) of F, one has

$$(29) \qquad (\lambda - \epsilon)\,(u'_j - u_j) \leqq F(\mathbf{Y}(\xi_j)\,;\ \mathbf{X}(u'_j) - \mathbf{X}(u_j)) < \mu(u'_j - u_j).$$

By adding the inequalities (29), one obtains

$$(30) \qquad (\lambda - \epsilon) \operatorname{meas} U \leqq \sum_{j=1}^{p} F(\mathbf{Y}(\xi_j)\,; \mathbf{X}(u'_j) - \mathbf{X}(u_j)) < \mu \operatorname{meas} U.$$

In virtue of (25) and the remark made immediately after Theorem II,

$$(31) \qquad \left|\, \sum_{j=1}^{p} F(\mathbf{Y}(\xi_j)\,;\ \mathbf{X}(u'_j) - \mathbf{X}(u_j)) - \int_U F(\mathbf{Y}; d\mathbf{X}) \,\right| < 2\epsilon.$$

Now

$$(32) \qquad \left| \int_T - \int_U \right| = \left| \int_{T-TU} - \int_{U-TU} \right| \leqq \left| \int_{T-TU} \right| + \left| \int_{U-TU} \right| < 2\epsilon$$

by (26) and the definition of η_ϵ. Finally, by (27), (30), (31), and (32),

$$(\lambda - \epsilon)(\operatorname{meas} T - \epsilon) - 4\epsilon \leqq \Phi\{T; \mathbf{X}\} < \mu(\operatorname{meas} T + \epsilon) + 4\epsilon.$$

The inequality (24) now follows since $\epsilon > 0$ is arbitrary.

Proof of Theorem III. Consider a lower and an upper Lebesgue approximating sum for the function $F(\mathbf{Y}(t)\,; \mathbf{X}'(t))$. Then, by Lemma 1,

$$(33) \qquad \sum_{i=-\infty}^{+\infty} \lambda_i \operatorname{meas} S\{\lambda_i \leqq F < \lambda_{i+1}\} \leqq \sum_{i=-\infty}^{+\infty} \Phi\{S_i\,; \mathbf{X}\} = \Phi\{[a, b)\,; \mathbf{X}\}$$

and

$$(34) \qquad \sum_{i=-\infty}^{+\infty} \lambda_i \text{ meas } S\{\lambda_{i-1} < F \leqq \lambda_i\} \geqq \sum_{i=-\infty}^{+\infty} \Phi\{S'_i; \mathbf{X}\} = \Phi\{[a,b); \mathbf{X}\}$$

where $S_i = S\{\lambda_i \leqq F < \lambda_{i+1}\}$ is the set of points t in $[a,b]$ for which $\lambda_i \leqq F(Y(t); X'(t)) < \lambda_{i+1}$ and $S'_i = S\{\lambda_{i-1} < F \leqq \lambda_i\}$ has an analogous meaning. The inequalities (33) and (34) imply that the integral occurring on the right of (22) exists and that (22) holds.

By Theorem II, if $\mathbf{X}(t)$ is purely discontinuous then $\Phi\{S; \mathbf{X}\}$ is a purely discontinuous set function. In this case, also, the integral has a simple form.

THEOREM IV. *If the assumptions of Theorem I are satisfied and if* $\mathbf{X}(t)$ *is purely discontinuous, then*

$$\int_a^b F(Y; d\mathbf{X}) = \sum F(\mathbf{Y}(\rho_i); \mathbf{X}(\rho_i + 0) - \mathbf{X}(\rho_i - 0)),$$

where the sum extends over all discontinuity points ρ_i *of* $\mathbf{X}(t)$.

Proof. By the remark immediately following the proof of Theorem I, it is seen that

$$(35) \qquad \int_a^b F(\mathbf{Y}; d\mathbf{X}) = \int_{\Sigma\rho_i} F(\mathbf{Y}; d\mathbf{X}) = \Sigma \int_{\rho_i} F(\mathbf{Y}; d\mathbf{X}).$$

But, by definition,

$$(36) \qquad \int_{\rho_i} F(\mathbf{Y}; d\mathbf{X}) = \lim_{\epsilon \to 0} \int_{\rho_i - \epsilon}^{\rho_i + \epsilon} F(\mathbf{Y}; d\mathbf{X})$$
$$= \lim_{\epsilon \to 0} F(\mathbf{Y}(\rho_i); \mathbf{X}(\rho_i + \epsilon) - \mathbf{X}(\rho_i - \epsilon)).$$

Theorem IV is a consequence of these relations.

4. Variational orthogonality. Two functions $x_1(t)$, $x_2(t)$ of bounded variation on $a \leqq t \leqq b$ will be said to be variationally orthogonal on $[a,b]$ if there exists a Borel set S such that

$$(37_1) \quad V\{S; x_1\} = V\{[a,b]; x_1\}; \qquad (37_2) \quad V\{S; x_2\} = 0,$$

where, as above, $V\{S; x\} = \int_S |dx|$ is the total variation of the function x on the set S. (It is clear that the seeming lack of symmetry of this definition with respect to x_1 and x_2 is only apparent.) For example, if $x_1(t)$ is purely singular and if $x_2(t)$ is absolutely continuous, then x_1 and x_2 are variationally orthogonal and S may be chosen to be a zero set satisfying (37_1). Similarly, if $x_1(t)$ is purely discontinuous and if $x_2(t)$ is continuous, then x_1 and x_2 are variationally orthogonal and S may be chosen to be the enumerable

set of points of discontinuity of $x_1(t)$. Finally let $x(t)$ be any function of bounded variation and let

$$(38)\qquad x_1(t)=\int_a^t \tfrac{1}{2}\{|\,dx\,|+dx\}$$

be the positive variation of x on $[a,t)$ and

$$(39)\qquad x_2(t)=x_1(t)-x(t)=\int_a^t \tfrac{1}{2}\{|\,dx\,|-dx\},$$

the negative variation of x on $[a,t)$, so that,

$$(40)\qquad x(t)=x_1(t)-x_2(t)$$

is Jordan's decomposition of $x(t)$ into the difference of two non-decreasing functions. Then $x_1(t)$ and $x_2(t)$ are variationally orthogonal and S may be chosen to be the set where $x'(t)\geqq 0$ (including $x'(t)=+\infty$).

The two vector functions

$$(41)\qquad \mathbf{X}_1(t)=\{x_{11}(t),\cdots,x_{1n}(t)\}\quad\text{and}\quad \mathbf{X}_2(t)=\{x_{21}(t),\cdots,x_{2n}(t)\}$$

will be said to be variationally orthogonal if $x_{1j}(t)$ and $x_{2j}(t)$ are variationally orthogonal for every j. The two vector functions (41) will be said to be completely variationally orthogonal if there exists a Borel set S for which

$$(42_1)\qquad V\{S;\mathbf{X}_1\}=V\{[a,b];\mathbf{X}_1\};\qquad (42_2)\qquad V\{S;\mathbf{X}_2\}=0,$$

i.e., for $j=1,\cdots,n$,

$$(43_1)\qquad V\{S;x_{1j}\}=V\{[a,b];x_{1j}\};\qquad (43_2)\qquad V\{S;x_{2j}\}=0.$$

Thus, if $\mathbf{X}_1(t)$ is purely singular and $\mathbf{X}_2(t)$ is absolutely continuous, then $\mathbf{X}_1(t)$ and $\mathbf{X}_2(t)$ are completely variationally orthogonal and S may be chosen to be a zero set satisfying (42_1). The same remarks hold if $\mathbf{X}_1(t)$ is purely discontinuous and $\mathbf{X}_2(t)$ is continuous. However, if $\mathbf{X}(t)$ is an arbitrary vector function of bounded variation and if $\mathbf{X}(t)=\mathbf{X}_1(t)-\mathbf{X}_2(t)$, represents the decomposition (38), (39), (40) of its components into monotone parts, then $\mathbf{X}_1(t)$ and $\mathbf{X}_2(t)$ are variationally orthogonal but need not be completely so.

5. Linearity. The property of complete variational orthogonality discussed above is exactly what is needed to insure that the integral (4) satisfies the decomposition property exhibited in (13).

THEOREM V. *Suppose that* (4) *exists absolutely and that* F *satisfies condition* (E). *Let* $\mathbf{X}(t)=\mathbf{X}_1(t)+\mathbf{X}_2(t)$, *where* $\mathbf{X}_1$ *and* $\mathbf{X}_2$ *are completely variationally orthogonal. Then*

$$(44)\qquad \int_a^b F(\mathbf{Y}; d\mathbf{X}) = \int_a^b F(\mathbf{Y}; d\mathbf{X}_1) + \int_a^b F(\mathbf{Y}; d\mathbf{X}_2),$$

the existence of the last two integrals being proved.

Proof. Since the conditions of Theorem I are satisfied, (16) exists and is a completely additive set function.

Let S denote the set satisfying (42). Let $\epsilon > 0$ be fixed. Let $U = U_\epsilon = \sum_{j=1}^{\infty} [u_j, u'_j)$ be an enumerable set of pairwise disjoint, half-open intervals which contains S. By the definition of $\Phi\{S; \mathbf{X}\}$, it is clear that U may be chosen so that

$$(45)\qquad | \Phi\{S; \mathbf{X}\} - \Phi\{U_\epsilon; \mathbf{X}\} | < \epsilon;$$

while from (42_2),

$$(46)\qquad V\{U_\epsilon; \mathbf{X}_2\} = \sum_{k=1}^{n} V\{U_\epsilon; x_{2k}\} < \delta_\epsilon,$$

where δ_ϵ is defined in E. Now, since Φ, V are completely additive set functions,

$$(47)\qquad \Phi\{U_\epsilon; \mathbf{X}\} = \sum_{j=1}^{\infty} \Phi\{[u_j, u'_j)\,;\, \mathbf{X}\}; \quad V\{U_\epsilon; \mathbf{X}\} = \sum_{j=1}^{\infty} V\{[u_j, u'_j)\,;\, \mathbf{X}\}.$$

Thus, it is possible to select a sufficiently large integer $N = N_\epsilon$, such that

$$(48)\qquad \left| \sum_{j=1}^{N} \Phi\{[u_j, u'_j)\,;\, \mathbf{X}\} - \Phi\{U_\epsilon; \mathbf{X}\} \right| < \epsilon,$$

and at the same time,

$$(49)\qquad \sum_{j=N+1}^{\infty} \sum_{k=1}^{n} V\{[u_j, u'_j)\,;\, x_{1k}\} < \delta_\epsilon.$$

Let $a = t_0 < t_1 < \cdots < t_{p+1} = b$ be a mesh on $[a, b]$ such that each u_j and each u'_j, for $j = 1, \cdots, N$, occurs among the t_k. Let the degree of fineness Δ of this mesh be less than the number Δ_ϵ defined in (19) and (20). It is clear that the requirement that u_j and u'_j occur among the t_k is no essential restriction on the generality of this mesh. Let Σ'_k denote the sum over those k-values for which the interval $[t_k, t'_k)$ lies in an interval $[u_j, u'_j)$, $j \leqq N$. Let Σ''_k denote the sum over all remaining k-values. Then

$$(50)\qquad \left| \sum_{k=0}^{p} F(\mathbf{Y}(\xi_k)\,;\, \Delta_k\mathbf{X}_1) - \Sigma'_k F(\mathbf{Y}(\xi_k)\,;\, \Delta_k\mathbf{X}) \right|$$
$$\leqq | \Sigma'_k \{F(\mathbf{Y}(\xi_k)\,;\, \Delta_k\mathbf{X}_1) - F(\mathbf{Y}(\xi_k)\,;\, \Delta_k\mathbf{X})\} | + |\Sigma''_k F(\mathbf{Y}(\xi_k)\,;\, \Delta_k\mathbf{X}_1)|.$$

Now, by (46),

$$(51)\qquad \Sigma'_k \sum_{j=1}^{n} | \Delta_k x_{1j} - \Delta_k x_j | = \Sigma'_k \sum_{j=1}^{n} | \Delta_k x_{2j} | \leqq \sum_{j=1}^{n} V\{U_\epsilon; x_{2j}\} < \delta_\epsilon,$$

and, by (49) and (42_1),

$$(52) \qquad \Sigma''_k \sum_{j=1}^{n} | \Delta_k x_{1j} | \leqq \sum_{j=N+1}^{\infty} \sum_{k=1}^{n} V\{[u_j, u'_j) ; x_{1k}\} + V\{[a, b] - U ; x_{1k}\} < \delta_\epsilon.$$

It follows, from (50), (51), (52) and the property (E) of F, that

$$(53) \qquad \left| \sum_{k=0}^{p} F(\boldsymbol{Y}(\xi_k) ; \Delta_k \boldsymbol{X}_1) - \Sigma'_k F(\boldsymbol{Y}(\xi_k) ; \Delta_k \boldsymbol{X}) \right| < 2\epsilon.$$

In virtue of the remark made following Theorem II and the fact that $\Delta < \Delta_\epsilon$,

$$(54) \qquad \left| \Sigma'_k F(\boldsymbol{Y}(\xi_k) ; \Delta_k \boldsymbol{X}) - \sum_{j=1}^{N} \Phi\{u_j, u'_j) ; \boldsymbol{X}\} \right| < 2\epsilon.$$

Addition of the inequalities (45), (48), (53), and (54) gives

$$(55) \qquad \left| \sum_{k=0}^{p} F(\boldsymbol{Y}(\xi_k) ; \Delta_k \boldsymbol{X}_1) - \Phi\{S ; \boldsymbol{X}\} \right| < 6\epsilon \text{ if } \Delta < \Delta_\epsilon.$$

Since the mesh employed is arbitrary, (55) shows that

$$(56) \qquad \int_a^b F(\boldsymbol{Y} ; d\boldsymbol{X}_1) = \Phi\{S ; \boldsymbol{X}\}$$

exists. In exactly the same way, it can be shown that

$$(57) \qquad \int_a^b F(\boldsymbol{Y} ; d\boldsymbol{X}_2) = \Phi\{[a, b] - S ; \boldsymbol{X}\}$$

exists. Since Φ is an additive set function, (56) and (57) together imply (44). This completes the proof of Theorem V.

The proof of Theorem V implies

THEOREM VI. *Suppose that* (4) *exists absolutely and that F satisfies condition (E'). Let $\boldsymbol{X}(t) = \boldsymbol{X}_1(t) + \boldsymbol{X}_2(t)$, where $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ are completely variationally orthogonal. Then*

$$\int_a^b F(\boldsymbol{Y} ; d\boldsymbol{X}) = \int_a^b F(\boldsymbol{Y} ; d\boldsymbol{X}_1) + \int_a^b F(\boldsymbol{Y} ; d\boldsymbol{X}_2),$$

whenever the last two integrals exist.

In fact, condition (E) was used in the proof of Theorem V only in connection with the inequalities (50), (51), (52), (53) and it is obviously possible to use condition (E′) in the same way if the existence of $\int_a^b F(\boldsymbol{Y} ; d\boldsymbol{X}_1)$ and $\int_a^b F(\boldsymbol{Y} ; d\boldsymbol{X}_2)$ is assured.

THEOREM VII. *Suppose that* (4) *exists and that F satisfies conditions*

(B), (C), and (D). Let $\mathbf{X}(t) = \mathbf{X}_1(t) + \mathbf{X}_2(t)$, *where* $\mathbf{X}_1$ *and* $\mathbf{X}_2$ *are completely variationally orthogonal. Then*

$$\int_a^b F(\mathbf{Y}; d\mathbf{X}) = \int_a^b F(\mathbf{Y}; d\mathbf{X}_1) + \int_a^b F(\mathbf{Y}; d\mathbf{X}_2),$$

the existence of the last two integrals being proved.

Proof. In virtue of Theorem V, it is sufficient to prove that conditions (B), (C), and (D) imply condition (E) and the existence of (15). Now, by (C),

$$F(\mathbf{Y}(t); \mathbf{Z}) - F(\mathbf{Y}(t); \mathbf{W}) \leqq F(\mathbf{Y}(t); \mathbf{Z} - \mathbf{W})$$

and

$$F(\mathbf{Y}(t); \mathbf{W}) - F(\mathbf{Y}(t); \mathbf{Z}) \leqq F(\mathbf{Y}(t); \mathbf{W} - \mathbf{Z}).$$

Thus, by (B),

$$\begin{aligned} &| F(\mathbf{Y}(t); \mathbf{Z}) - F(\mathbf{Y}(t); \mathbf{W}) | \\ &\qquad \leqq F(\mathbf{Y}(t); \mathbf{Z} - \mathbf{W}) = F(\mathbf{Y}(t); \mathbf{W} - \mathbf{Z}). \end{aligned} \tag{58}$$

This inequality (58) shows, first of all, that $F(\mathbf{Y}(t); \mathbf{Z}) \geqq 0$ for all $\mathbf{Z}$, so that existence coincides with absolute existence. Also (58) and the condition (D) imply (E).

THEOREM VIII. *Suppose that* (4) *exists and that* F *satisfies conditions (B), (C) and (D′). Let* $\mathbf{X}(t) = \mathbf{X}_1(t) + \mathbf{X}_2(t)$, *where* $\mathbf{X}_1$ *and* $\mathbf{X}_2$ *are completely variationally orthogonal. Then*

$$\int_a^b F(\mathbf{Y}; d\mathbf{X}) = \int_a^b F(\mathbf{Y}; d\mathbf{X}_1) + \int_a^b F(\mathbf{Y}; d\mathbf{X}_2),$$

whenever the last two integrals exist.

Proof. Examination of the proof of Theorems V and VI shows that condition (E′) is required only in the following forms,

$$\left| \sum_{j=1}^{\infty} \{F(\mathbf{Y}(\xi_j); \Delta_j\mathbf{X}) - F(\mathbf{Y}(\xi_j); \Delta_j\mathbf{X}_1)\} \right| < \epsilon \text{ if } \sum_{j=1}^{\infty} \sum_{k=1}^{n} | \Delta_j x_{2k} | < \delta_\epsilon,$$

$$\left| \sum_{j=1}^{\infty} \{F(\mathbf{Y}(\xi_j); \Delta_j\mathbf{X}) - F(\mathbf{Y}(\xi_j); \Delta_j\mathbf{X}_2)\} \right| < \epsilon \text{ if } \sum_{j=1}^{\infty} \sum_{k=1}^{n} | \Delta_j x_{1k} | < \delta_\epsilon.$$

Thus, the required formulae corresponding to (58) are

$$\begin{aligned} &| F(\mathbf{Y}(\xi); \Delta\mathbf{X}) - F(\mathbf{Y}(\xi); \Delta\mathbf{X}_1) | \leqq F(\mathbf{Y}(\xi); \Delta\mathbf{X}_2) \\ &| F(\mathbf{Y}(\xi); \Delta\mathbf{X}) - F(\mathbf{Y}(\xi); \Delta\mathbf{X}_2) | \leqq F(\mathbf{Y}(\xi); \Delta\mathbf{X}_1). \end{aligned} \tag{59}$$

These formulae (59) are, obviously, consequences of (B) and (C). But (59) shows that the form of (E′) which is actually used in the proof of Theorem VI is implied by (D′). Thus Theorem VIII is a consequence of the proof of Theorem VI.

COROLLARY 1. *Let $X(t) = X_1(t) + X_2(t) + X_3(t)$, where $X_1(t)$ is absolutely continuous, $X_2(t)$ is purely singular, and $X_3(t)$ is purely discontinuous. Then, under the assumptions of any of Theorems V-VIII,*

$$(60)\qquad \int_a^b F(Y;dX) = \int_a^b F(Y;dX_1) + \int_a^b F(Y;dX_2) + \int_a^b F(Y;dX_3).$$

It is understood that the existence of all three integrals on the right of (60) is either supposed or proved as in the corresponding theorem.

6. Variationally orthogonal decompositions. It has been seen that if the vector $X(t)$ is decomposed into the sum of completely variationally orthogonal vectors then, in a large number of cases, (4) is correspondingly decomposed linearly. In case $X(t)$ is decomposed into the sum of two vectors which are variationally orthogonal but not necessarily completely so, the behavior of (4) is described by the following theorems.

THEOREM IX. *Suppose that $\int_a^b F(Y;dX) = \int_a^b F(Y;dx_1,\cdots,dx_n)$ and $\int_a^b F(Y;0,dx_2,\cdots,dx_n)$ exist absolutely. Suppose that F satisfies the following condition: For every $\epsilon > 0$, there exists a $\delta_\epsilon > 0$, such that*

$$(61)\qquad \left|\sum_{j=1}^{\infty}\{F(Y(\xi_j);\, z_{j1}, z_{j2},\cdots,z_{jn}) - F(Y(\xi_j);\, w_{j1}, z_{j2},\cdots,z_{jn})\}\right| < \epsilon \text{ if } \sum_{j=1}^{\infty}|z_{j1}-w_{j1}| < \delta_\epsilon.$$

Let $x_1(t) = x_{11}(t) + x_{21}(t)$, where x_{11}, x_{21} are variationally orthogonal. Then

$$(62)\qquad \int_a^b F(Y;dX) = \int_a^b F(Y;dx_{11},dx_2,\cdots,dx_n) + \int_a^b F(Y;dx_{21},dx_2,\cdots,dx_n) - \int_a^b F(Y;0,dx_2,\cdots,dx_n),$$

the existence of the first two integrals on the right being proved.

Theorem IX may be proved by an argument which is a modification of the proof of Theorem V and which will not be given here. It is found that the modification (61) of the condition (E) is all that is required in this case. A similar modification of the conditions (B), (C), (D), (D′), and (E′), in such a way that they apply only to the particular variable under consideration yield theorems which are analogous to Theorems VI, VII, VIII in the same way as Theorem IX is analogous to Theorem V.

Repeated applications of Theorem IX to successive variables yield

THEOREM X. *Suppose that* (4) *exists absolutely. Suppose that F satisfies condition* (E). *Let $\mathbf{X}(t) = \mathbf{X}_1(t) + \mathbf{X}_2(t) + \mathbf{X}_3(t)$, where $\mathbf{X}_1$ and $\mathbf{X}_2$ are variationally orthogonal vectors and $\mathbf{X}_3 \equiv 0$. Then*

$$(63)\qquad \int_a^b F(\mathbf{Y}; d\mathbf{X}) = \sum_{i_1=1}^{3} \cdots \sum_{i_n=1}^{3} (-1)^{k(i_1,\cdots,i_n)} \int_a^b F(\mathbf{Y}; dx_{i_1 1}, \cdots, dx_{i_n n}),$$

(*where* $k(i_1, \cdots, i_n) = \sum_{j=1}^{n} [i_j/3] =$ *number of values j for which* $i_j = 3$)

provided all the integrals occurring on the right of (63) *exist absolutely.*

7. The Hellinger integral.[4] The Hellinger integral is the special case of (4) for which $F = z_1^2/ \mid z_2 \mid$ so that the integral has the form

$$(64)\qquad \Phi(\mathbf{X}) = H_1(x_1, x_2) = \int_a^b (dx_1)^2/ \mid dx_2 \mid.$$

In dealing with these integrals it is always assumed that $x_2(t)$ is monotone non-decreasing in the interval $[a, b]$, so that the absolute value sign in the denominator of (64) is unnecessary, but it has been included here in order to make F satisfy condition (B). It is assumed that every interval of constancy of $x_2(t)$ is an interval of constancy of $x_1(t)$. Finally, the undetermined fraction 0/0 is defined to be 0.

The following known properties of (64) will be needed.

(i)[5] In order that (64) exist, it is necessary and sufficient that there exist a function $h(t)$, monotone non-decreasing in $[a, b]$, such that, if $a \leqq t < t' \leqq b$, then

$$(65)\qquad [x_1(t') - x_1(t)]^2 \leqq [x_2(t') - x_2(t)][h(t') - h(t)],$$

or, symbolically,

$$(65)\qquad (\Delta x_1)^2 \leqq (\Delta x_2)(\Delta h).$$

(ii)[6] Let $x_1(t) = g_1(t) - g_2(t)$ represent the decomposition (38), (39), (40) of $x_1(t)$ into the difference of two non-decreasing, variationally orthogonal functions. Then, if (64) exists, $H_1(g_1, x_2)$ and $H_1(g_2, x_2)$ exist and

$$H_1(x_1, x_2) = H_1(g_1, x_2) + H_1(g_2, x_2).$$

[4] Cf. Hellinger [4], Hahn [2], and Svenson [8].

[5] Cf. Hellinger [4], p. 26 f. or Svenson [8], p. 4.

[6] Svenson [8], Theorem IV, p. 29. This statement is also a consequence of one of the modifications of Theorem IX above and, in fact, holds for any decomposition of $x_1(t)$ into variationally orthogonal parts.

(iii)[7] If $x_1(t)$ is non-decreasing and if (64) exists, then the function $h(t)$ in (i) may be chosen so that $x_1(t)$ is a basis[8] of $h(t)$.

Now it will be shown that Theorems I, II, III, IV, and VIII are applicable to the Hellinger integral by proving

LEMMA 2. *The Hellinger integral* (64) *satisfies conditions* (*B*), (*C*) *and* (*D'*).

Proof. Since conditions (B) and (C) are obviously satisfied by $F = (z_1)^2/|z_2|$ it is only required to prove that (D') also is satisfied.

Let $X(t) = \{x_1(t), x_2(t)\}$ be a vector function for which (64) exists. In view of (ii) it is sufficient to treat the case that $x_1(t)$, as well as $x_2(t)$, is non-decreasing. In this case, (i) and (iii) show that there exists a non-decreasing function $h(t)$, of which $x_1(t)$ is a basis, satisfying (65). Now, it is known[9] that if $x_1(t)$ is a basis of $h(t)$, then for every $\epsilon > 0$, there exists a $\delta_\epsilon > 0$ such that

$$\sum_{j=1}^{\infty} \Delta_j h < \epsilon \quad \text{whenever} \quad \sum_{j=1}^{\infty} \Delta_j x_1 < \delta_\epsilon.$$

Now suppose that $\sum_{j=1}^{\infty} [\Delta_j x_1 + \Delta_j x_2] < \delta_\epsilon$. Then,

$$\sum_{j=1}^{\infty} (\Delta_j x_1)^2/\Delta_j x_2 \leqq \sum_{j=1}^{\infty} \Delta_j h < \epsilon,$$

since $\Delta_j x_1 < \Delta_j x_1 + \Delta_j x_2$. But this is exactly condition (D').

Thus it is seen, in particular, that the Hellinger integral is additive under completely variationally orthogonal decompositions $\mathbf{X} = \mathbf{X}_1 + \mathbf{X}_2$, whenever $\Phi(\mathbf{X}_1)$ and $\Phi(\mathbf{X}_2)$ exist. It will now be shown that this is always the case.

LEMMA 3. *Suppose that* (64) *exists where* $x_i(t)$, $i = 1, 2$, *are non-decreasing. Let* $x_i(t) = x_{1i}(t) + x_{2i}(t)$, *where* $x_{ij}(t)$, $i, j = 1, 2$, *are non-decreasing and where the vectors* $\mathbf{X}_1 = \{x_{11}, x_{12}\}$ *and* $\mathbf{X}_2 = \{x_{21}, x_{22}\}$ *are completely variationally orthogonal. Then* $H_1(x_{11}, x_{12})$ *and* $H_1(x_{21}, x_{22})$ *exist.*

Proof. Since $H_1(x_1, x_2)$ exists, by (i), there exists a monotone non-decreasing function $h(t)$ satisfying (65). By (65) and Schwarz's inequality

$$\Sigma\Delta_j x_1 \leqq \Sigma(\Delta_j x_2)^{\frac{1}{2}}(\Delta_j h)^{\frac{1}{2}} \leqq (\Sigma\Delta_j x_2)^{\frac{1}{2}}(\Sigma\Delta_j h)^{\frac{1}{2}},$$

so that, if S is any Borel set,

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[7] Svenson [8], pp. 11-12.

[8] A monotone function $b(t)$ is called a basis of the function $f(t)$ if $V\{S; f\} = 0$ whenever $V\{S; b\} = 0$. Thus, $h(t)$ is clearly a basis of $x_1(t)$ by (65). Cf. Radon [6], p. 1318.

[9] Cf. Radon [6], p. 1319.

$$(66)\qquad [V\{S;x_1\}]^2 \leqq V\{S;x_2\}V\{S;h\}.$$

Now by the definition (43) of complete variational orthogonality, there exists a set T such that

$$V\{T;x_{1i}\} = V\{[a,b);\ x_{1i}\} \quad \text{and} \quad V\{T;x_{2i}\} = 0.$$

Thus, if $[t,t')$ is any half-open interval in $[a,b)$,

$$(67)\qquad V\{[t,t');\ x_{1i}\} = V\{[t,t')T;x_{1i}\}$$

and

$$(68)\qquad V\{[t,t')T;x_i\} = V\{[t,t')T;x_{1i}\}.$$

In particular, (67) and (68) give

$$V\{[t,t');\ x_{11}\} = V\{[t,t')T;x_1\},$$

or, using (66),

$$(69)\qquad [V\{[t,t');x_{11}\}]^2 \leqq V\{[t,t')T;x_2\}V\{[t,t')T;h\}.$$

But, by (68),

$$(70)\qquad V\{[t,t')T;x_2\} = V\{[t,t')T;x_{12}\}.$$

Together, (69) and (70) imply

$$[V\{[t,t');x_{11}\}]^2 \leqq V\{[t,t');\ \ x_{12}\}V\{[t,t');\ h\}$$

or

$$(\Delta x_{11})^2 \leqq (\Delta x_{12})(\Delta h).$$

Hence, by (i), $H(x_{11},x_{12})$ exists. The existence of $H(x_{21},x_{22})$ is shown similarly.

Some of the results of this section are summarized, for the particular case of the Lebesgue decomposition, in

COROLLARY 2. *Suppose that* (64) *exists. Let*

$$x_i(t) = a_i(t) + s_i(t) + p_i(t), \qquad (i=1,2),$$

where a_i is absolutely continuous, s_i is purely singular, and p_i is purely discontinuous. Then

$$H_1(x_1,x_2) = H_1(a_1,a_2) + H_1(s_1,s_2) + H_1(p_1,p_2)$$

or

$$\int_a^b (dx_1)^2/dx_2 = \int_a^b [(x'_1(t))^2/x'_2(t)]dt + \int_a^b (ds_1)^2/ds_2$$
$$+ \Sigma[x_1(\rho_i+0) - x_1(\rho_i-0)]^2/[x_2(\rho_i+0) - x_2(\rho_i-0)],$$

where the last sum is taken over all discontinuity points ρ_i of $x_1(t)$.

It may be mentioned that the results of this section also apply to the generalisation,[10] $\int_a^b |dx_1|^p / |dx_2|^{p-1}$, $p > 1$, of Hellinger's integral.

8. The Hilbert integral.[11] The Hilbert integral is the special case of (1) for which $F = |z_1 z_2|^{\frac{1}{2}}$, so that the integral has the form

$$\Phi(X) = H_2(x_1, x_2) = \int_a^b |dx_1 dx_2|^{\frac{1}{2}}. \tag{71}$$

This integral exists whenever $X(t) = \{x_1(t), x_2(t)\}$ is of bounded variation. It will now be shown that Theorems I, II, III, IV and VI are applicable to this integral by proving

LEMMA 4. *The Hilbert integral* (71) *satisfies condition* (E').

Proof. Let $X_i(t) = \{x_{i1}(t), x_{i2}(t)\}$, $i = 1, 2$, be vectors of bounded variation, so that $H_2(x_{11}, x_{12})$ and $H(x_{21}, x_{22})$ exist. Let $M = \max_{i,j=1,2} V\{[a, b]; x_{ij}\}$. Now, from

$$|a + b|^{\frac{1}{2}} |c + d|^{\frac{1}{2}} \leqq |ac|^{\frac{1}{2}} + |ad|^{\frac{1}{2}} + |bc|^{\frac{1}{2}} + |bd|^{\frac{1}{2}},$$

one obtains

$$\begin{aligned} |\Delta x_{11}\Delta x_{12}|^{\frac{1}{2}} - |\Delta x_{21}\Delta x_{22}|^{\frac{1}{2}} \leqq{}& |\Delta x_{12} - \Delta x_{22}|^{\frac{1}{2}} |\Delta x_{21}|^{\frac{1}{2}} \\ &+ |\Delta x_{11} - \Delta x_{21}|^{\frac{1}{2}} |\Delta x_{22}|^{\frac{1}{2}} + |\Delta x_{12} - \Delta x_{22}|^{\frac{1}{2}} |\Delta x_{11} - \Delta x_{21}|^{\frac{1}{2}}, \end{aligned} \tag{72}$$

by letting $\Delta x_{11} = a + b$, $\Delta x_{12} = c + d$, $\Delta x_{21} = a$ and $\Delta x_{22} = c$. Using (72) and the Schwarz inequality, one has

$$\sum_{j=1}^{\infty} \{|\Delta_j x_{11}\Delta_j x_{12}|^{\frac{1}{2}} - |\Delta_j x_{21}\Delta_j x_{22}|^{\frac{1}{2}}\} \leqq 4M^{\frac{1}{2}}\{\sum_{j=1}^{\infty}\sum_{k=1}^{2} |\Delta_j x_{1k} - \Delta_j x_{2k}|\}^{\frac{1}{2}} \tag{73}$$

where $\Delta_j x_{ik} = x_{ik}(t'_j) - x_{ik}(t_j)$ and $\{[t_j, t'_j)\}$ is any set of non-overlapping half-open intervals on $[a, b]$.

By repeating the above argument with the subscripts 1 and 2 interchanged, it is seen that (73) still holds if the absolute value of the first sum is taken. Thus

$$\left|\sum_{j=1}^{\infty} \{|\Delta_j x_{11}\Delta_j x_{12}|^{\frac{1}{2}} - |\Delta_j x_{21}\Delta_j x_{22}|^{\frac{1}{2}}\}\right| < \epsilon \text{ if } \sum_{j=1}^{\infty}\sum_{k=1}^{2} |\Delta_j x_{1k} - \Delta_j x_{2k}| < \delta_\epsilon, \tag{74}$$

where $\delta_\epsilon = \epsilon^2/16M$. But (74) is precisely condition (E').

Recalling the fact that condition (E') implies condition (D') it is seen that the theorems mentioned above are applicable to the Hilbert integral (71). Again restating the particular case of the Lebesgue decomposition, one has

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[10] F. Riesz [6].

[11] Hellinger [4], p. 31 f.

COROLLARY 3. *Let $x_i(t) = a_i(t) + s_i(t) + p_i(t)$, $i = 1, 2$, where a_i is absolutely continuous, s_i is purely singular, and p_i is purely discontinuous. Then*

$$H_2(x_1, x_2) = H_2(a_1, a_2) + H_2(s_1, s_2) + H_2(p_1, p_2)$$

or

$$\int_a^b |\, dx_1 dx_2 \,|^{\frac{1}{2}} = \int_a^b |\, x'_1(t) x'_2(t) \,|^{\frac{1}{2}} dt + \int_a^b |\, ds_1 ds_2 \,|^{\frac{1}{2}}$$
$$+ \Sigma \,|\, x_1(\rho_i + 0) - x_1(\rho_i - 0) \,|^{\frac{1}{2}} \,|\, x_2(\rho_i + 0) - x_2(\rho_i - 0) \,|^{\frac{1}{2}},$$

where the last sum is taken over all common discontinuity points ρ_i of $x_1(t)$ and $x_2(t)$.

9. Riemannian arc length. The length of curves in Riemannian spaces is given by the integral

$$(75) \qquad \int_a^b \left\{ \sum_{i,k=1}^{n} g_{ik}(x_1, \cdots, x_n)\, dx_i dx_k \right\}^{\frac{1}{2}},$$

where $g_{ik}(t) = g_{ik}(x_1(t), \cdots, x_n(t))$ is the matrix of a positive definite quadratic form. It is assumed that $g_{ik}(t)$ is continuous on $[a, b]$. Then the integral (75) exists whenever $\mathbf{X}(t) = \{x_1(t), \cdots, x_n(t)\}$ is of bounded variation. It will be shown that Theorems I, II, III, IV, VII, IX and X are applicable to these integrals (75) by proving

LEMMA 5. *The integral* (75) *satisfies conditions* (*B*), (*C*) *and* (*D*).

Proof. First, the condition (B) is obviously satisfied.

To see that condition (D) is satisfied, let $M = \max |\, g_{ik}(t) \,|$ for $a \leqq t \leqq b$, $i, k = 1, \cdots, n$. Then

$$\left\{ \sum_{i,k=1}^{n} g_{ik}(t) z_i z_k \right\}^{\frac{1}{2}} \leqq M^{\frac{1}{2}} \sum_{i=1}^{n} |\, z_i \,|.$$

Finally, the inequality required in (C) is known [12] to hold in this case.

COROLLARY 4. *Let $x_i(t) = a_i(t) + s_i(t) + p_i(t)$, $i = 1, \cdots, n$, where a_i is absolutely continuous, s_i is purely singular and p_i is purely discontinuous. Then*

$$(76) \qquad \int_a^b \left\{ \sum_{i,k=1}^{n} g_{ik}(\mathbf{X})\, dx_i dx_k \right\}^{\frac{1}{2}} = \int_a^b \left\{ \sum_{i,k=1}^{n} g_{ik}(\mathbf{X})\, da_i da_k \right\}^{\frac{1}{2}}$$
$$+ \int_a^b \left\{ \sum_{i,k=1}^{n} g_{ik}(\mathbf{X})\, ds_i ds_k \right\}^{\frac{1}{2}} + \int_a^b \left\{ \sum_{i,k=1}^{n} g_{ik}(\mathbf{X})\, dp_i dp_k \right\}^{\frac{1}{2}}.$$

[12] This inequality is equivalent to inequality 29 in Hardy, Littlewood and Pólya [3], p. 33.

10. Euclidean arc length. The ordinary Euclidean arc length is, of course, the special case of the above for which $g_{ik} = \delta_{ik}$. However, the results obtained take such a simple form, in this case, that it seems worthwhile to enumerate them. For simplicity of statement, these results will be formulated in the case $n = 2$.

COROLLARY 5. *Let $x_i(t) = a_i(t) + s_i(t) + p_i(t)$, $i = 1, 2$, where a_i is absolutely continuous, s_i is purely singular, and p_i is purely discontinuous. Then*

$$(77)\quad \int_a^b \{(dx_1)^2 + (dx_2)^2\}^{\frac{1}{2}} = \int_a^b \{(x'_1(t))^2 + (x'_2(t))^2\}^{\frac{1}{2}} dt$$
$$+ \int_a^b \{(ds_1)^2 + (ds_2)^2\}^{\frac{1}{2}} + \Sigma\{[x_1(\rho_i + 0) - x_i(\rho_i - 0)]^2$$
$$+ [x_2(\rho_i + 0) - x_2(\rho_i - 0)]^2\}^{\frac{1}{2}},$$

where the last sum is taken over all discontinuity points ρ_i of $x_1(t)$ or of $x_2(t)$.

COROLLARY 6. *Suppose $\boldsymbol{S}(t) = \{s_1(t), s_2(t)\}$ is a continuous vector function of bounded variation, such that, for every t on $[a, b]$, either $s'_1(t) = 0$ or $s'_2(t) = 0$. Then*

$$\int_a^b \{(ds_1)^2 + (ds_2)^2\}^{\frac{1}{2}} = V\{[a, b)\,;\ s_1\} + V\{[a, b)\,;\ s_2\}.$$

In fact, in the decomposition $\boldsymbol{S} = \boldsymbol{S}_1 + \boldsymbol{S}_2$, where $\boldsymbol{S}_1(t) = \{s_1(t), 0\}$ and $\boldsymbol{S}_2(t) = \{0, s_2(t)\}$, the two vectors $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ are completely variationally orthogonal.

COROLLARY 7. *The length of the curve $y = y(x)$, where $y(x)$ is a continuous function of bounded variation in $[a, b]$, is given by*

$$\int_a^b \{1 + (y'(t))^2\}^{\frac{1}{2}} dt + V\{[a, b)\,;\ y\} - \int_a^b |\, y'(t) \,|\, dt.$$

This is a consequence of Corollary 5, for in this case $a_1(t) = t$, $a_2(t) = \int_a^t y'(u)\,du$, $s_1(t) = 0$, $s_2(t) = y(t) - a_2(t)$.

COROLLARY 8. *The length of the curve $y = y(x)$, where $y(x)$ is a purely singular monotone function in $[a, b]$ is given by*

$$|\, y(b) - y(a) \,| + |\, b - a \,|.$$

COROLLARY 9. *Let $x_1(t) = g(t) - h(t)$ be the decomposition* (38), (39), (40) *of $x_1(t)$ into the difference of two monotone non-decreasing, variationally orthogonal functions. Then*

$$\int_a^b \{(dx_1)^2 + (dx_2)^2\}^{\frac{1}{2}} = \int_a^b \{(dg)^2 + (dx_2)^2\}^{\frac{1}{2}} + \int_a^b \{(dh)^2 + (dx_2)^2\}^{\frac{1}{2}} - V\{[a, b); x_2\}.$$

This corollary is a consequence of Theorem VIII.

COROLLARY 10. *Let $x_i(t) = g_i(t) - h_i(t)$ be the decomposition of $x_i(t)$ into the difference of two monotone non-decreasing variationally orthogonal functions. Then*

$$\begin{aligned}\int_a^b \{(dx_1)^2 + (dx_2)^2\}^{\frac{1}{2}} = \int_a^b & \{(dg_1)^2 + (dg_2)^2\}^{\frac{1}{2}} \\ & + \int_a^b \{(dh_1)^2 + (dh_2)^2\}^{\frac{1}{2}} + \int_a^b \{(dg_1)^2 + (dh_2)^2\}^{\frac{1}{2}} \\ & + \int_a^b \{(dh_1)^2 + (dg_2)^2\}^{\frac{1}{2}} \\ & - V\{[a, b); x_1\} - V\{[a, b); x_2\}.\end{aligned}$$

This corollary is a consequence of Theorem IX.

In Corollaries 9 and 10 it is clear that the variational orthogonality of g_i and h_i is all that is needed to insure the result.

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ON TRANSLATIONS IN GENERAL PLANE GEOMETRIES.*

By Herbert Busemann.

In a well-known paper, Hilbert [1] has characterized the Euclidean and hyperbolic plane geometries by mere group and continuity axioms. He gets all the motions *at once* by requiring the existence of sufficiently many *rotations.* The present paper tries to point out how the existence of more and more *translations gradually* specializes the rather general metric it starts with to a Desarguesian geometry (Minkowskian or hyperbolic).

We require our initial space to be a "Geradenraum" in Menger's terminology.[2] The exact definition will be found in section 1. It is essentially a metric space with exactly one shortest line (s. l.) of infinite length passing through two given points. We therefore call a Geradenraum an SL space. In order to be able to formulate the results, we must have the concept of an asymptote. If g is any shortest line, P a point not on g, and Q a point traversing g in a certain direction $\overrightarrow{g}$ ($\overleftarrow{g}$), the s. l. connecting P and Q always tends to a limit s. l., which we call an *asymptote* to $\overrightarrow{g}$, preserving the word *parallel* for the case where the two asymptotes to $\overrightarrow{g}$ and $\overleftarrow{g}$ through P coincide. Section 1 gives those properties of these asymptotes and of limit circles which we shall need later on.

A two-dimensional SL space Σ will be called a *plane* since it is homeomorphic to the Euclidean plane.[3] As usually, we say the metric of Σ is *Desarguesian* if it is possible to map Σ topologically on the Euclidean plane or a convex part K of it in such manner that the shortest lines are transformed into straight lines or into the intersections of straight lines with K.[4] By a motion of Σ we mean any one-to-one mapping of Σ onto itself which preserves distance and, in particular, by a *translation along a shortest line* g we mean a motion which transforms each of the half planes of Σ defined by g into itself.

We first assume that to each pair A, B of points on a fixed s. l. g there

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[1] Reprinted in [5], Anhang IV. The numbers [n] refer to the references on p. 256.

[2] [7] especially pp. 100-113.

[3] See [2].

[4] A proof for the fact that the validity of Desargues' Theorem is necessary and sufficient for the existence of such a mapping, can be found in [11] §§ 3, 4.

exists a translation along g carrying A into B; these translations form a group G. In section 2 we ascertain which of the usual properties of translations hold, and show by examples that others do not. The main result is that *the images of a fixed point A under the translations of G form a curve which together with g bounds a convex domain.* It follows then, for instance, that a parallel h to g is equidistant from g and that the translations along g can also be regarded as translations along h; but such a geometry is not necessarily Desarguesian, even if the parallel axiom holds throughout the plane. On the other hand, it can be Desarguessian with the parallel axiom holding only with respect to g.

The availability of G does not imply the existence of translations along a non-parallel asymptote to $\overleftarrow{g}$ (or $\overrightarrow{g}$). By assuming their presence one therefore gets a much more specialized metric (section 3), in which translations along each asymptote a' to $\overleftarrow{g}$ exist and *the hyperbolic formula for the arc of the limit circle to the direction $\overleftarrow{g}$ between a' and g holds.* Nevertheless an example will show that the geometry is *not necessarily Desarguesian.*

However, *as soon as one requires the existence of translations along two s. l. where the one is neither an asymptote nor parallel to the other, we get a Desarguesian metric.* We find, namely, (section 4) that *the metric is either Minkowskian*[5] *or hyperbolic.* In both cases there exist translations along each straight line.

1. Asymptotes and limit spheres in SL spaces. The exact definition of an SL space is as follows: A complete metric space with the distance function $r(x, y)$ is an SL space if it satisfies the conditions:

I. To each pair of points A, B there exists exactly one point C, the center of A and B for which $r(A, C) + r(C, B) = r(A, B)$, $r(A, C) = \dfrac{r(A, B)}{2}$.

II. To each pair of distinct points A, B there exist exactly two points D and D' such that B is the center of A and D and A the center of B and D'.

All points X satisfying the relation

$$r(A, X) + r(X, B) = r(A, B)$$

form a point set (designated by $\overline{AB}$ or $\overline{BA}$), which is homeomorphic to an Euclidean straight line segment.[6] If $X \subset \overline{AB}$ and $X \neq A$, $X \neq B$ we say

[5] The original definition is in [8], chapter I, compare also p. 234 of this paper.

[6] *loc. cit.* [2].

that X is between A and B. The set of points Y (resp. Z) for which B is between A and Y (resp. A between B and Z) together with $\overline{AB}$ is homeomorphic to a Euclidean ray and will be designated by $\underrightarrow{AB}$, resp. $\underrightarrow{BA}$. We put

$$\underline{AB} = \underrightarrow{AB} + \underrightarrow{BA}$$

and call $\underline{AB}$ a *shortest line* (*s. l.*). $\underline{AB}$ is homeomorphic to a Euclidean straight line.[6] Then the following theorem holds:

(1.1) *Through two different points passes exactly one* s. l. *Hence two different* s. l. *intersect in at most one point.*

As usually we put for any two point sets

$$r(\alpha, \beta) = \underset{X \subset \alpha,\ Y \subset \beta}{\text{G. l. b.}}\, r(X, Y)$$

We say that the point sets α_ν converge to the set α if α contains all limit points of sequences $\{P_\nu\}$ with $P_\nu \subset \alpha_\nu$ and if to any point O, any positive number r, and any $\epsilon > 0$, a number $N(\epsilon, O, r)$ can be found such that for $\nu > N$ and each point $X \subset \alpha$ with $r(X, O) < r$ the inequality

$$r(X, \alpha_\nu) < \epsilon$$

holds. Then we have

(1.2) *If $A_\nu \to A$ and $B_\nu \to B$, then $\overline{A_\nu B_\nu} \to \overline{AB}$; and if $A \neq B$, also*

$$\underrightarrow{A_\nu B_\nu} \to \underrightarrow{AB}, \qquad \underrightarrow{B_\nu A_\nu} \to \underrightarrow{BA}, \qquad \underline{A_\nu B_\nu} \to \underline{AB}.$$

If, for a given point P, a point $Q < \alpha$ exists with

$$r(P, Q) = r(P, \alpha)$$

we call Q a *foot* of P on α. Then the following lemma is an immediate consequence of the triangle inequality.

(1.3) *If $R_\nu \to R$, $\alpha_\nu \to \alpha$, and T_ν is a foot of R_ν on α_ν, then each accumulation point T of T_ν is a foot of R on α.*

Let g be any s. l., $P \not\subset g$, $A \subset g$, and X a variable point on g. $r(X, A) \to \infty$ implies $r(P, X) \to \infty$; hence, P has at least one foot Q on g. A point R between P and Q has Q as its only foot on g; for, from

$$r(R, Q') \leq r(R, Q), \qquad Q' \subset g$$

would follow

$$r(P,Q') < r(P,R) + r(R,Q) = r(P,g).$$

We shall have to use later the fact:

(1. 4) *In an* SL *space of dimension greater than* 1, *there exist to each s. l. g points whose distance from g is arbitrarily great.*

To prove this we show that a sphere with radius a whose center C is on g[7] contains points P with $r(P,g) \geq a/2$. Let B_1, B_2 be the points on g with $r(B_1,C) = r(B_2,C) = a$, B'_1 the center of $\overline{B_1C}$, B'_2 the center of $\overline{B_2C}$ and A an arbitrary point not on g. We draw $\overline{B_1A}$ and $\overline{B_2A}$ and consider the set σ formed by all rays $\overrightarrow{CX}$ with $X \subset \overline{B_1A} + \overline{AB_2}$. Let K be the intersection of the sphere with σ; K is homeomorphic to a Euclidean semicircle. The feet on g of a point $P \subset K$ near B_i are near B_i $(i = 1, 2)$. If one had $r(P,g) < a/2$ for all $P \subset K$ no foot on g of a point $P \subset K$ could belong to $\overline{B'_1B'_2}$. It is easy to see that there must be a point P_0 with two feet Q_1, Q_2 on g such that $Q_1 \subset \overrightarrow{B'_1B_1}$, $Q_2 \subset \overrightarrow{B'_2B_2}$ and one would have

$$a > r(P_0,Q_1) + r(P_0,Q_2) \geq r(Q_1,Q_2) \geq a.$$

(1. 4), together with the preceding statement, gives:

(1. 5) *In an* SL *space of dimension greater than* 1, *to a given s. l. g and a given* $N > 0$ *points P can be found with a unique foot on g and* $r(P,g) = N$.

We now consider an oriented s. l. g and write $Q < Q'$ for two points Q and Q' on g if Q' follows Q. If $P < Q < Q'$ it follows from the triangle inequality that

$$K(Q,P) \cdot K(Q',P) = P \tag{1.6}$$

($\alpha \cdot \beta$ means the product of the point sets α and β) where $K(X,Z)$ means the sphere with the center X and through Z, i. e. with the radius $r(X,Z)$. More exactly, $K(P,Q) - P$ lies in the interior of $K(P,Q')$. From this we conclude: if $P < Q_1 < Q_2 < \cdots$ and $r(P,Q_\nu) \to \infty$, the spheres $K(Q,P)$ tend to a limit set $L(P,g)$, which does not depend on the choice of the sequence $\{Q_\nu\}$.[8] We call $L(P,g)$ the *limit sphere through P with or to the center ray g*. In terms of the metric of the space, $L(P,g)$ can be characterized as follows: it consists of those points R for which

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[7] By the sphere with radius a and center C we mean the set of points Z with $r(Z,C) = a$.

[8] For detailed proofs of this and the following statements see [3] § 2.

(1.7) $$r(Q_\nu, P) - r(Q_\nu, R) \to 0$$

for each sequence $Q_1 < Q_2 < \cdots$ with $r(P, Q) \to \infty$.

One proves with the help of (1.7) that the limit spheres with the center ray g cover the space. $L(P, g)$ decomposes the space into two parts, the interior and the exterior of $L(P, g)$, the former consisting of those points which, for $P < Q$ and sufficiently large $r(P, Q)$, are in the interior of $K(Q, P)$.

We call the point set μ equidistant from the point set ν if, for any two points P_1, P_2 of μ one has

$$r(P_1, \nu) = r(P_2, \nu) = r(\mu, \nu).$$

If μ is equidistant from ν, then ν is not necessarily equidistant from ν. But for limit spheres one has

(1.8) *Limit spheres with the same center ray g are equidistant from each other.* Hence exactly one limit sphere with g as center ray passes through a given point.

With the help of these results we can prove a lemma which will be applied frequently in the sequel.

(1.9) Lemma. *If the s. l. s and a intersect in T then the parts of a and s outside any sphere around T with positive radius have positive distance.*

Let us assume, on the contrary, that there exists a sequence $\{R^*_\nu\}$ on a and a sequence $\{S^*_\nu\}$ on s, such that

$$r(R^*_\nu, S^*_\nu) \to 0 \quad \text{but} \quad \begin{array}{l} r(R^*_\nu, T) > \delta > 0 \\ r(S^*_\nu, T) > \delta > 0. \end{array}$$

Then we must have $r(R^*_\nu, T) \to \infty$.

Let A be a point on a different from T. Either on $\overrightarrow{AT}$ or on $\overrightarrow{TA}$ are infinitely many points R_ν of $\{R^*_\nu\}$; suppose they are on $\overrightarrow{TA}$, and call $\{S_\nu\}$ the subsequence of $\{S^*_\nu\}$ corresponding to $\{R_\nu\}$. The two limit spheres $L(A, \overrightarrow{TA})$ and $L(A, \overrightarrow{AT})$ have no common interior point. Since $r(S_\nu, R_\nu) \to 0$, the s. l. s intersects both these limit spheres, the former at B, say, and the latter at B', with $B' \subset \overline{TB}$. On account of (1.6) the sphere $K(T, A)$ intersects the segment $\overline{TB'}$ in a point B'' between T and B'; hence, $r(T, B) > r(T, A)$. On the other hand, we have $| r(B, S_\nu) - r(B, R_\nu) | < r(R_\nu, S_\nu)$ and

$$r(B, R_\nu) - r(A, R_\nu) \to 0 \qquad \text{(see (1.7))}.$$

Hence

$$r(B, S_\nu) - [r(A, R_\nu) + r(R_\nu, S_\nu)] \to 0$$

and

$$r(T, S_\nu) - [r(T, R_\nu) + r(R_\nu, S_\nu)] \to r(T, B) - r(T, A) > 0,$$

which for large ν contradicts the triangle inequality.

From reference [3], section 2, we take the following theorem:

(1.10) *An arbitrary point A has a unique foot on each limit sphere. The feet of A on the different limit spheres with the same center ray g form a s. l. a.*

We call a the *asymptote* to g through A. Two asymptotes to g are either identical or disjoined. In order to justify the use of this term we show

(1.11) *If $\{Q^*_\nu\} < g$, $Q^*_1 < Q^*_2, \cdots$, and $r(Q^*_1, Q^*_\nu) \to \infty$, then $\underline{AQ^*_\nu}$ tends to the asymptote a to g through A.*

Proof. Let $L(P, g)$ be the limit sphere with the center ray g through A, and $\{Q_\nu\}$ a subsequence of $\{Q^*_\nu\}$ for which $\underline{AQ_\nu}$ converges to a s. l. b. (That $\{Q_\nu\}$ exists follows from (1.2)). We have to prove that b coincides with a. For $r(A, Q_\nu) > 1$ we can choose the points R_ν and T_ν on $\overline{AQ_\nu}$ such that $r(A, R_\nu) = 1$ and $r(Q_\nu, P) = r(Q_\nu, T_\nu)$ (that $r(Q_\nu, A) > r(Q_\nu, P)$ follows from (1.1)). The points R_ν tend to a point R on b with $r(A, R) = 1$, R is not on $L(P, g)$. T_ν is the foot of R on $K(Q_\nu, P)$. We therefore conclude from (1.3) and $K(Q_\nu, P) \to L(P, \vec{g})$ that each limit point of $\{T_\nu\}$ must be a foot of R on $L(P, g)$. (1.10) shows that R has only one foot T on $L(P, g)$, namely the point where the asymptote $\underline{RT}$ to g through R intersects $L(P, g)$. Therefore,

$$b = \lim \underline{QA} = \lim \underline{RT} = \underline{RT}$$

Hence b is an asymptote to g and from $b \cdot a \supset A$ follows $b \equiv a$.

We can formulate our results thus:

(1.12) *If Q traverses the s. l. g in a certain direction $\vec{g}$ and $P \not\subset g$, then $\underline{PQ}$ tends to a limit s. l. a, the asymptote to $\vec{g}$ through P. The asymptote to $\vec{g}$ through a point of* a *coincides with* a. *A converging sequence of asymptotes to $\vec{g}$ tends to an asymptote to $\vec{g}$.*

From now on we suppose that our SL space, indicated by Σ, has dimension 2. It is therefore hemeomorphic to the Euclidean plane.[3] The spheres

are homeomorphic to Euclidean circles; we therefore speak of *circles* and *limit circles* instead of spheres and limit spheres. Through each point A not on a s. l. g we have two (possibly coincident) asymptotes to g. They determine two angles, an open one and closed one. The former is determined by the property that it contains all the s. l. through A intersecting g and no others; the latter is the complement to the former. If the two asymptotes coincide, the closed angle consists of exactly one s. l., which we call a *parallel* to g.

Let $\vec{a}$ be an asymptote to $\vec{g}$. The question arises as to whether the family of the limit circles to $\vec{a}$ is (as in the hyperbolic and Euclidean geometries) always identical with that of the limit circles to $\vec{g}$. We shall see that only part of this is true:

(1.13) *Let π^* be that one of the two half-planes defined by a which is contained in one of the half-planes determined by g, π^{**} the other one. Then,* for each $P \subset \vec{g}$.

$$L(P,\vec{g})\pi^* \equiv L(T,\vec{a})\pi^* \qquad (T = L(P,\vec{g}) \cdot a),$$

and $L(T,\vec{a})\pi^{**}$ is in the interior or on $L(P,\vec{g})$.

To prove this, we first state that for each pair of distinct points A, B and each pair of positive numbers a, b, with $a + b > r(A,B)$, exactly two points T and T', one on each side of $\underline{AB}$, exist such that

(1.14) $\qquad r(A,T) = r(A,T') = a, \qquad r(B,T) = r(B,T') = b.$[9]

If $b = a$, we can say, furthermore, that $\overline{TT'}$ lies in the interior of the quadrangle $ATBT'$. For, otherwise, we would have

$$r(T,A) = r(T,B), \qquad r(T',A) = r(T',B)$$

with A and B on the same side of $\underline{TT'}$.

We now consider any point P of g. $L(P,\vec{g})$ may intersect a in T. Let A be any point following P on g, and B the point on a with $r(B,T) = r(A,P)$. Except for the point T, the circle $K(B,T)$ lies in the interior of $L(P,\vec{g})$, since T is the unique foot of B on $L(P,\vec{g})$. This statement contains the second part of (1.13). On account of (1.14) the circles $K(A,P)$ and $K(B,T)$ intersect exactly twice for sufficiently large $r(A,P)$. Since $K(A,P)$

[9] Compare [2].

is (except for P) in the interior of $L(P, \vec{g})$, one of the intersections, for large $r(A, P)$, must be between a and g near the arc $\widehat{PT}$ of $L(P, \vec{g})$, and the other outside of an arbitrarily large circle around T. Hence, for large $r(A, P)$, an arbitrary great portion of $K(B, T) \cdot \pi^*$ lies between $K(A, P)$ and $L(P, \vec{g})$, which proves the remainder of the assertion.

It follows that:

(1.15) *The asymptotes to* $\vec{a}$ *in* π^* *are also asymptotes to* $\vec{g}$.
This can, of course, be easily seen without using (1.13).

We now show by an example that, even in a Desarguesian geometry with the Euclidean parallel axiom it can happen that in the above notation $L(T, \vec{a})\pi^{**}$ lies wholly in the interior of $L(P, \vec{g})$.

We recall the definition of a Minkowskian geometry: Let $r = \lambda(\phi)$ be a convex curve in the strict sense in the Cartesian (r, ϕ) or (x, y)-plane with $r = 0$ as center. We define the distance between two points $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$ to be the number

$$\bar{r}(P_1, P_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}\,\frac{1}{\lambda(\phi)}$$

where $+\phi$ or $-\phi$ is the direction of the straight line $\underline{P_1P_2}$, i. e.

$$\phi = \pm \operatorname{arc} tg \frac{y_2 - y_1}{x_2 - x_1}.$$

As shown by Minkowski[5] $\bar{r}(P_1, P_2)$ satisfies all our conditions and the whole Euclidean straight lines are the s. l. of our space. Now let $r = \lambda(\phi)$ be the curve

$$x^2 + \left(y + \frac{1}{\sqrt{2}}\right)^2 = 1 \quad \text{for } y \geq 0,$$

$$x^2 + \left(y - \frac{1}{\sqrt{2}}\right)^2 = 1 \quad \text{for } y \leq 0.$$

The limit circles to center rays with directions different from 0 and π are straight lines. The limit circle to the positive x-axis through 0 is the curve

$$(*) \qquad \begin{cases} y = x & \text{for } y \geq 0, \\ y = -x & \text{for } y \leq 0. \end{cases}$$

The limit circle to a center ray $y = \beta$ parallel to the positive x-axis through the point (α, β) is obtained from $(*)$ by a translation which carries $(0, 0)$ into (α, β). One sees: if $\beta > 0$ and $\alpha = \beta$ (for $\beta < 0$, $\alpha = -\beta$) the

two limit circles are identical only for $y \geq \beta$ $(y \leq \beta)$ and the second is in the interior of (*) for $y < \beta$ $(y > \beta)$.

I do not know if the concept of an asymptote is always symmetric, i. e. if from the fact that a is an asymptote to g always follows that g is an asymptote to a. A sufficient condition is that the limit spheres to a are also limit spheres to g, but this condition is not necessary, as we have just seen. Another sufficient condition is given by

(1.16) *If the distance between two non-intersecting s. l.'s a and b vanishes, then a is an asymptote to b and b to a.*

For, there exists a sequence of points R_ν on b, tending to infinity in a certain direction, say $\vec{b}$, and a sequence of points S on a with $r(S_\nu, R_\nu) \to 0$. We select an arbitrary point A on a. The s. l. $\underline{AR_\nu}$ tend to the asymptote c to b through A. If c is not identical with a, it must intersect $\overline{S_\nu R_\nu}$ in a point S'_ν and we would have $r(S_\nu, S'_\nu) \to 0$ in contradiction to (1.9).

The converse to (1.16) cannot always hold, since it is not true in Euclidean geometry. But one could conjecture that it is true if the parallel axiom of hyperbolic geometry holds throughout the plane, especially in a Desarguesian geometry in a bounded part of the Euclidean plane. To show that this *is not so,* we recall the definition of a certain geometry introduced by Hilbert.[10] Let K be any bounded, closed, convex curve in the plane. The distance between two points A, B interior to K is defined as follows: Let $\overrightarrow{AB}$ intersect K in Y and $\overrightarrow{BA}$ in X. Designating the Euclidean distance by $e(\ )$ we put

$$r^*(A, B) = \log \frac{e(A, Y) \cdot e(B, X)}{e(B, Y) \cdot e(A, X)}.$$

Hilbert proves that $r^*(A, B)$ satisfies our conditions if K is convex in the strict sense and that otherwise the straight lines are shortest lines but not necessarily the only s. l. Therefore

$$r^*(A, B) + e(A, B)$$

is, for each bounded, convex curve K, a distance function which defines an SL space. Choosing K as a triangle, one sees that the s. l. issuing from a vertex of K are asymptotes to each other, but the distance between any two distinct ones among these s. l. is positive. One should note that the distance of the "other ends" of two such s. l. is finite.

[10] See [5], Anhang I.

2. Geometries with a group of translations along one shortest line. Let g be a s. l. in the two dimensional SL space Σ, π_1 and π_2 the two half-planes into which g decomposes Σ. A one-to-one transformation of Σ into itself, which preserves the distances of corresponding pairs of points and transforms π_1, π_2 into themselves will be called a *translation of* Σ *along* g. Such a translation transforms a s. l. into a s. l.

To a given pair of points A, B on g there exists at most one translation along g carrying A into B. For let C be any point on g different from A, then any translation γ along g carrying A into B transforms C into the point $D c g$ with $\gamma(C,D) = \gamma(A,B)$ and either

$$\underset{\longrightarrow}{AB} \supset \underset{\longrightarrow}{CD} \quad \text{or} \quad \underset{\longrightarrow}{CD} \supset \underset{\longrightarrow}{AB}.$$

Since (compare 1. 14) a point X of π_1 is uniquely determined by the two distances $r(X,A)$ and $r(X,C)$, γ must carry X into the unique point $X' \subset \pi_1$ with $r(X',B) = r(X,A)$ and $r(X',D) = r(X,C)$. This proves our statement.

A translation γ along g, therefore, is uniquely determined by the fact that it carries the first of a pair of points A, B on g into the second. We designate γ by $(A \to B)$. It follows that

$$(A \to B)(B \to C) = (A \to C) \qquad (\text{first } (A \to B))$$

and since the transformations of g into itself induced by translations along g are commutative, the translations along g are commutative. We now assume that to each pair of points A, B on g the translation $(A \to B)$ exists. Then these translations form an Abelian group G. Hereafter, we indicate the assumption of the presence of G by saying that *all translations along* g *exist.* We first notice some simple consequences of the existence of G.

(2. 1) *Translations different from the identity have no fixed points.*

If P remained fixed under $(A_0 \to A_1) \neq 1$, P would also be fixed under the positive powers $(A_0 \to A_2)$, $(A_0 \to A_3)$, $\cdots$ of $(A_0 \to A_1)$. We should have

$$r(A_0,P) = r(A_1,P) = r(A_2,P) = \cdots;$$

but
$$r(A_0,A_\nu) = \nu \cdot r(A_0,A_1) \to \infty.$$

(2. 2) *Each point P has a unique foot on g.*

Assume that $P \subset \pi_1$ has two different feet F_1 and F_2 on g. The proof of (1. 5) shows that we can choose a point Q in π_1 with a unique foot F on G and $r(Q,g) > r(P,F_1) + r(P,F_2)$. Let F' be any point between F_1 and F_2.

$(F \to F')$ transforms Q into a point Q' which has F' as unique foot on g. Moreover Q' cannot lie in the interior of the triangle PF_1F_2. Hence $\overline{QF'}$ must intersect $\overline{F_1P} + \overline{PF_2}$ in a point R, which would have two different feet on g, contrary to page 230.

(2.3) *The points of Σ having the same point F of g as foot form a s. l., which we call the perpendicular to g in F.*

This is always true as soon as each point of Σ has exactly one foot on g, i. e. the existence of G is unessential. For if $P_1 \subset \pi_1$ and $P_2 \subset \pi_2$, both have F as foot, $\overline{P_1P_2}$ must intersect g in F. If $Q \neq F$ were the intersection, we should have

$$r(P_1, F) + r(F, P_2) > r(P_1, P_2) = r(P_1, Q) + r(Q, P_2)$$

and either

$$r(P_1, F) > r(P_1, Q) \quad \text{or} \quad r(F, P_2) > r(Q, P_2).$$

If, now, P_3 is any other point, say in π_1, with F as foot, then for the same reason P_3 must be on $\underline{P_2Q}$ $(= \underline{P_1Q})$.

As an immediate consequence we have

(2.4) *$(A \to B)$, $A, B \subset g$, transforms the perpendicular p to g at A into the perpendicular q to g at B.*

The point on p in π_1 which has distance r from A is carried into the point in π_1 on q which has distance r from B. We consider all the points which have distance r from g. They form two curves c_r^1, c_r^2, the one in π_1, the other in π_2. c_r^1 and c_r^2 are equidistant from g and are transformed into themselves by each translation along g. We now prove the important fact:

(2.5) *The curves c_r^1 and c_r^2 are convex; that is, a suitable one of the sides of $c_r^1(c_r^2)$ (called the interior of $c_r^1(c_r^2)$) together with $c_r^1(c_r^2)$ has the property of containing a s. l. segment $\overline{AB}$ if it contains A and B.*

We consider c_r^1 for some $r > 0$. If c_r^1 contains a s. l. segment it is a s. l., since it is transformed into itself by each motion of G.

We therefore assume that c_r^1 contains no s. l. segment. Let A', B' be any points of c_r^1, $\widehat{A'_1B'}$ the corresponding arc of c_r^1. If $\underline{A'B'}$ intersects $\widehat{A'B'}$ in more points than A' and B' we can find a subarc $\widehat{AB}$ of $\widehat{A'B'}$ such that $\widehat{AB} \cdot \underline{AB} = A + B$. Let π be the half plane determined by $\underline{AB}$ which contains

$\widehat{AB} - A - B$, and D any point of $AB - \overline{AB}$. The perpendicular a to g through D does not intersect $\widehat{AB}$, because each perpendicular intersects $c_r{}^1$ exactly once and $\widehat{AB}$ is met by the perpendiculars through the points of $\overline{AB}$. Let s be a ray of π^* issuing from D. We vary s continuously from the position $a \cdot \pi^*$ towards $\overrightarrow{DB}$. $\widehat{AB}$ being bounded, there exists a first ray s^0 having common points with $\widehat{AB}$, $s^0 \neq \overrightarrow{DB}$. Let T be any point of $\widehat{AB} \cdot s^0$. Using again the fact that each perpendicular to g intersects $c_r{}^1$ exactly once one sees that there exists a circular disc with center T such that the (open) half λ_T, which is on the same side of s^0 as $a \cdot \pi^*$ has no common points with $c_r{}^1$.

Let T_1 be any point of $c_r{}^1$. The translation along g carrying T into T_1 transforms λ_T into a congruent semi-circular disc with center T_1 disjoined from $c_r{}^1$ and on the same side of $c_r{}^1$ as λ_T. Calling this side the exterior and the other the interior of $c_r{}^1$, a theorem of Tietze [11] shows that the interior of $c_r{}^1$ plus $c_r{}^1$ is convex. It is true that Tietze assumes the metric to be Euclidean, but the simple proof for Tietze's theorem given by Reinhardt [12] can be carried over to our case without any change whatsoever.

From this proof it follows that at each point of $c_r{}^1$ there exists a supporting line (this can be proved quite generally for convex curves in arbitrary two-dimensional SL spaces). For, since $c_r{}^1$ contains no segment and the s. l. $\bar{s}$ bearing s^0 certainly contains an exterior point of $c_r{}^1$, the s. l. $\bar{s}$ must be, except for T, completely in the exterior of $c_r{}^1$. By translations of $\bar{s}$ along g we get supporting lines at each point of $c_r{}^1$. It is also easy to see that $\bar{s}$ is the only supporting line of $c_r{}^1$ at T such that $c_r{}^1$ has at each point a unique tangent. But we do not need this later on. We determine now which side of $c_r{}^1$ is the interior by proving

(2.6) *The domain bounded by $c_r{}^1$ ($c_r{}^2$) and g is convex.*

Using T and $\bar{s}$ with the same meaning as previously, at least one, say $\bar{s}'$, of the two rays on $\bar{s}$ issuing from T does not intersect g. Let the perpendicular to g through T cut g at Q and let Q_1 be a point on g on the same ("right") side of the perpendicular as $\bar{s}'$. If $r(Q, Q_1) > 0$ is sufficiently small the translation $(Q \to Q_1)$ transforms $\bar{s}$ into an s. l. $\bar{s}_1$ intersecting $\bar{s}$ in a point L of $\bar{s}'$. If T_1 is the image of T under $(Q \to Q_1)$, the ray $\overrightarrow{LT_1}$ contains the image of $\bar{s}'$. $c_r{}^1$ intersects all perpendiculars to g. Therefore, if the assertion were not true, i. e. if $\bar{s}$ and $\bar{s}_1$ were except for T respectively T_1 between $c_r{}^1$

[11] In [12]. [12] In [10].

and g, the ray $\overrightarrow{LT_1}$ would have to intersect all perpendiculars to g on the right side of the one through L. Let X_1 traverse $\overrightarrow{LT_1}$ and let X be the point where the perpendicular to g through X_1 intersects $\bar{s}'$. Since each curve c_r^1 is cut at most twice by $\bar{s}_1$ and $\bar{s}$, and these s. l. are between c_r^1 and g, the numbers $r(X_1, g)$ and $r(X, g)$ should converge to certain limits r_0^1 and r_0 with $0 \leq r_0^1 \leq r$ and $0 \leq r_0 \leq r$. Then we must have $r_0 = r'_0$ since $\bar{s}_1$ is the image of $\bar{s}$ under $(Q \rightarrow Q_1)$. Hence we should have $r(X, X') \rightarrow 0$ which contradicts our Lemma (1.9).

We conclude from (2.6) that an s. l. h intersecting g in a point A meets each curve c_r^1 or c_r^2 at most once and is therefore transformed by any translation $(A \rightarrow A')$ of G into an s. l. h' not intersecting h. For, if $h \cdot h' = S$, $(A \rightarrow A')$ would carry S into a point S' on h', A', S, S' would be on h' and S and S' would be on the same curve c_r^1 or c_r^2. We see, furthermore, that if X traverses one of the rays of L determined by A in one direction, $r(X, g)$ increases. Distinguishing $\overrightarrow{g}$ and $\overleftarrow{g}$ one finds: If Y traverses an asymptote a to $\overrightarrow{g}$ in such a direction, that the foot F_Y of Y traverses $\overrightarrow{g}$, the distance $r(Y, F_Y)$ either decreases monotonically in the strict sense or is constant for all Y on a. For, if $r(Y, F_Y)$ is constant on a certain segment of a this segment belongs to a curve c_r^i; then a coincides with this c_r^i. This leads to the following theorem:

(2.7) *A curve c_r^i is a shortest line if, and only if, it is parallel to g.*

For, let $p \neq g$ be a parallel to g in π_1. Our last statement shows that $r(Y, g)$ decreases or remains constant if Y traverses p in either direction; hence $r(Y, g)$ is constant and p is a curve c_r^1.

The converse is a little more involved. We show first: If c_r^1 is a s. l. and $(A \rightarrow B)$ transforms the point $\bar{A}$ of c_r^1 into the point $\bar{B}$, then

(2.8) $$r(A, B) = r(\bar{A}, \bar{B}).$$

Let A_0, B_0 be the feet of $\bar{A}, \bar{B}$ on g. We have

$$r(A_0, B_0) = r(A, B).$$

Call B_0^{n-1}, $\bar{B}^{n-1}$ the images of B_0 respectively B under $(A \rightarrow B)^n$. The points $\bar{B}, \bar{B}^1, \cdots, \bar{B}^{n-1}$ are on c_r^1. We have

$$r(A_0, B_0^{n-1}) = n \cdot r(A_0, B_0)$$
$$r(\bar{A}, \bar{B}^{n-1}) = n \cdot r(\bar{A}, \bar{B})$$

hence

$$n\cdot|r(A_0,B_0)-r(\bar{A},\bar{B})|=|r(A_0,B_0{}^{n-1})-r(\bar{A},\bar{B}^{n-1})|$$
$$<r(\bar{A},A_0)+r(\bar{B}^{n-1},B_0)=2r.$$

The same argument gives the more general fact: If $(A\to B)$ transforms $\bar{A}$ into $\bar{B}$, then

$$r(\bar{A},\bar{B})\geq r(A,B).$$

For, considering the images of $\bar{B}_0$ and B_0 under $(A\to B)$ we find, as above,

$$n\cdot r(\bar{A},\bar{B})+2r\geq n\cdot r(A,B).$$

We remark, furthermore, that the metric characterization (1.7) of the limit spheres yields:

(2.9) $(A\to B)$ *transforms the limit circle* $L(A,\overrightarrow{g})$ *into* $L(B,\overrightarrow{g})$ *and* $L(A,\overleftarrow{g})$ *into* $L(B,\overleftarrow{g})$. With the help of (2.8) and (2.9) we prove (2.7) as follows. If $c_r{}^1$ were an s. l. but not an asymptote, for instance, to $\overrightarrow{g}$, the asymptote a to $\overrightarrow{g}$ through a point P of $c_r{}^1$ would be different from $c_r{}^1$. The limit circle to $\overrightarrow{g}$ through P may cut g in P_0. Let $(P_0\to Q_0)$ be any translation in the direction $\overleftarrow{g}$. $(P_0\to Q_0)$ transforms $L(P_0,\overrightarrow{g})$ into $L(Q_0,\overrightarrow{g})$. Designating the intersections of $L(Q_0,\overrightarrow{g})$ with $c_r{}^1$ and a by Q and Q', we would have $r(P,Q')=r(P_0,Q_0)$ on account of (1.8) and $r(P,Q)=r(P_0,Q_0)$ as a consequence of (2.8), in contradiction to (1.10).

We see that a translation $(A\to B)$ along g is also a translation $(\bar{A}\to\bar{B})$ along $c_r{}^1$, with $r(A,B)=r(\bar{A},\bar{B})$, if $c_r{}^1$ is an s. l. and $\bar{A}<c_r{}^1$. (2.6) shows that then the curves $c_r{}^1$ with $0<r<r_0$ are also s. l. Putting our results together we find:

(2.10) *If* $c_{r_0}{}^1$, $r_0>0$, *is an s. l., then the curves* $c_r{}^1$ *with* $0\leq r\leq r_0$ *are also* s. l.; *all these* $c_r{}^1$ *are parallel to each other; and a translation* $(A\to B)$ *along* g *is at the same time a translation* $(A'\to B')$ *along any of the* $c_r{}^1$ *with* $r(A,B)=r(A',B')$. *An s. l. h intersecting* g *intersects all* $c_r{}^1$ *for* $0\leq r\leq r_0$. *If* h' *is the transform of* h *under* $(A\to B)$, $A,B\subset g$, *the* s. l. h *and* h' *cut out segments of the same length* $r(A,B)$ *on all these* $c_r{}^1$ *and two different* $c_r{}^1$ *cut out segments of equal length on* h *and* h'.

This could lead to the conjecture that if through each point of the plane a parallel to g exists (we say in this case: the parallel axiom holds with

respect to g), the parallel axiom must hold with respect to each s. l. But we show by an example that this is not necessarily so, not even in Desarguesian geometry. We introduce a metric in the part $|y| < 1$ of the (x, y)-plane by putting

$$r((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + \left| \log \frac{(y_1 - 1)(y_2 + 1)}{(y_1 + 1)(y_2 - 1)} \right| + \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

One sees easily that $r(\ )$ defines an SL space and that the parts of the Euclidean straight lines in $|y| < 1$ are the s. l. of the metric, which therefore is Desarguesian. Furthermore, the parallel axiom holds with respect to the x-axis and all translations along the x-axis exist. We remark that the s. l. $x =$ constant, $|y| < 1$, are equidistant from each other. The circles of this metric are in general not convex. A result of P. Funk[13] shows that it is impossible to introduce in $|y| < 1$ a (Desarguesian) metric with these s. l. for which the equidistant curves to each straight line are again straight lines. Section 4 of this paper implies that one cannot find both a metric with these s. l. and translations along two s. l. which are neither parallel nor asymptotes to each other.

We have seen that *the existence of G, the Desarguesian character of the metric, and the parallel axiom with respect to g do not imply the validity of the parallel axiom with respect to each s. l.* On the other hand, we are going to show by another example that the existence of G and the validity of the parallel axiom for each s. l. do not imply the Desarguesian character of the metric.

For, let Σ be the whole Euclidean plane and let g be the x-axis. We introduce a Minkowskian metric $r_1(P, Q)$ in $\pi_1 + g$ $(y \geq 0)$ and a different one, $r_2(P, Q)$ in $\pi_2 + g (y \leq 0)$ but so that the diameters parallel to g of the unit circles both have the Euclidean length 1. Then one has ($e(\ )$ is the Euclidean distance) $e(P, Q) = r_1(P, Q) = r_2(P, Q)$ for $P, Q \subset g$. Let $P_1 = (x_1, y_1)$, $y_1 > 0$ be any point π_1 and $P_2 = (x_2, y_2)$, $y_2 < 0$ any point of π_2. Then the functions $r_1(P_1, (x, 0))$ and $r_2(P_2, (x, 0))$ are convex; hence their sum is convex and there exists a uniquely determined point $X_0 \subset g$, such that

$$r(P_1, P_2) = \min_X [r_1(P_1, (x, 0)) + r_2(P_2, (x, 0))] = r(P_1, X_0) + r(P_2, X_0).$$

Putting

$$\text{(2.11)} \qquad \begin{aligned} r(P, Q) &= r_1(P, Q) \quad \text{if} \quad P, Q \subset \pi_1 + g \\ r(P, Q) &= r_2(P, Q) \quad \text{if} \quad P, Q \subset \pi_2 + g \end{aligned}$$

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[13] In [4].

defines $r(P, Q)$ for all points of the plane. One sees easily that $r(P, Q)$ satisfies all our conditions. The s. l. are the straight lines $y = \text{const.}$ and straight lines which are broken at a point of the x-axis. Parallel rays in $\pi_1 + g$ have parallel continuations in $\pi_2 + g$; therefore *the parallel axiom holds for each* s. l. *All translations along each line $y = const.$ exist. But the theorem of Desargues is not true,* if the circles of the two Minkowskian geometries have no special relations to each other, for instance, if the circles in π_1 are ellipses and those in π_2 are not.

Returning to the general case, suppose that there exists a last $r \geq 0$, say r_0, such that $c_{r_0}{}^1$ is an s. l. g_1, and let π^* be that half plane determined by g_1, which is contained in π_1. Then through each point of π^* there pass two different asymptotes to g_1. Let a be an asymptote to $\overleftarrow{g_1}$. From (2. 9) it follows that the images of a under the translations of G are again asymptotes to $\overleftarrow{g_1}$, and (2. 6) shows that a translation γ in the direction $\overrightarrow{g}$ carries c into an s. l. between a and g_1. Hence the positive powers of γ must carry a into a sequence of asymptotes to $\overleftarrow{g_1}$, which, according to (1. 11), converges to an asymptote $\bar{a}$ to $\overleftarrow{g_1}$. $\bar{a}$ must remain invariant under all translations of G; therefore $\bar{a}$ is a curve $c_r{}^1$; and since $c_{r_0}{}^1 = g_1$ is the last curve $c_r{}^1$ which is an s. l., we must have $\bar{a} = g_1$. This implies that each asymptote a to g_1 in π^* has distance 0 from g_1, and that $r(x, g_1) \to \infty$ if x traverses a in the direction $\overrightarrow{a}$. From (1. 15) and (1. 16) we have that $\overleftarrow{a}$ is also an asymptote to $\overleftarrow{g_1}$ and to each other asymptote to $\overleftarrow{g_1}$ in π^*. Let h be any s. l. in π^* which is no asymptote to g. Orienting h in the "same" way as g, one sees easily that there exists an s. l. a_1 between h and g which is an asymptote to $\overrightarrow{h}$ and $\overleftarrow{g_1}$ and an s. l. a_2, which is an asymptote to $\overleftarrow{h}$ and $\overrightarrow{g_1}$. Hence, if y tends to infinity on h in either direction, one has $r(y, g_1) \to \infty$. There exists exactly one pair of points $S \subset h$ and $R \subset g_1$ such that $r(R, S) = r(h, g_1)$. S is determined as the point where a curve $c_r{}^1$ touches h and R in the foot of S on g_1 and inversely.

The first question which arises in this connection is if it is actually possible that $g_1 \neq g$ or $0 < r_0 < \infty$. *In a Desarguesian geometry it certainly can not happen.* For, if one of two Euclidean straight line segments (unbounded ones admitted) is parallel to another in our sense, both of them must be whole straight lines. The only convex domains in the Euclidean plane which contain a whole straight line are the whole plane, a half-plane, and the part of

the plane between two parallel lines. *But there are non-Desarguesian geometries with* $0 < r_0 < \infty$.

In order to construct such a metric, let ζ be the interior of the unit circle of the (x, y)-plane. Introduce in ζ a hyperbolic metric $h(P, Q)$ in such a way that the straight-line segments in ζ become the s. l. Let ζ_1 be the part $y \leq 0$, ζ_2 the part $y \geq 0$ of ζ, and π^* the half-plane $y \leq 0$. We map π^* topologically on ζ_2 by associating $O = (0, 0)$ with itself and a point $Q \neq 0$ to that point $\bar{Q}$ on $\overrightarrow{OQ}$ for which

$$e(O, Q) = h(O, \bar{Q}) \quad (e(\ ) \text{ is the Euclidean distance}).$$

If $(\bar{P}, \bar{Q})$ are any points in ζ_2 and P, Q the points in π^* to which they correspond, we put

$$r(\bar{P}, \bar{Q}) = e(P, Q).$$

In this way we have introduced in ζ_2 a Euclidean metric (for which the s. l. are in general no Euclidean straight line segments) which on $0 < x < 1$, $y = 0$ coincides with the hyperbolic metric. For two points in ζ_1 we put

$$r(\bar{P}, \bar{Q}) = h(\bar{P}, \bar{Q}).$$

By applying the same method which led to the combination of the two different Minkowskian geometries, we define $r(P, Q)$ also for $P < \zeta_1$, $Q < \zeta_2$. We thus get a metric in ζ, which makes ζ an SL space, with translations along the s. l. $g_1 : x = 0$. Taking any parallel $g \neq g_1$ to g_1 in ζ_2 one sees easily that these s. l. g and g_1 satisfy the assumptions of our previous considerations.

3. Geometries with translations along an s. l. g and its asymptotes. The last example also shows that the existence of all translations along g_1 (or g) does not imply the existence of translations along an asymptote to g_1 (or g) which is not a parallel.

Our considerations of the general case (compare p. 242) did indicate that by translations along g_1 (or g) any asymptote $\vec{a} \neq \vec{g}_1$ to $\vec{g}_1$ in π^* can be carried into any other one. If there exist also translations along a, we conclude from this that there exist two different asymptotes to $\vec{g}_1$ through each point $P \not\subset g_1$. For, if $g \neq g_1$, g would be an asymptote to $\vec{a}$, since g is an asymptote to $\vec{g}_1$ and $\vec{g}_1$ to $\vec{a}$ (compare (1.15)). Then $r(a, g) = 0$, but $r(a, g) \geq r(g_1, g) > 0$. From $r(a, b) = 0$ for each asymptote $\vec{b}$ to $\vec{a}$ and

$r(g, c) = 0$ for each asymptote $\vec{c}$ to $\vec{g} = \vec{g}_1$, one sees that the asymptotes to $\vec{g}$ are asymptotes to each other; and, since by a translation along $a^{(g)}$ any asymptote to g in π_2 (π_1) can be transformed into $g^{(a)}$ there exist translations along each asymptote to g. We have found

(3.1) *If $\vec{a}$ is an asymptote to $\vec{g}$ not parallel to g and if all translations exist along g as well as along a, then all asymptotes to $\vec{g}$ are asymptotes to each other and all translations along each of these s. l. exist. If b is any of them, then the two asymptotes to b through any point P not on b are different.*

We shall now study in a little more detail the properties of a metric satisfying the assumption of (3.1). Let π_1 and π_2 again be the two half planes determined by g. At first all considerations will refer only to one of these half planes, say π_1. We therefore put $P + L(P, \overleftarrow{g})\pi_1 = L_1(P, \overleftarrow{g})$. We again designate by $c_r{}^1$ the curve equidistant from g in π_1 with $r(c_r{}^1, g) = r$. (2.9) shows that the arcs of the different limit circles $L(P, \overleftarrow{g})$ between any two fixed curves $c_{r_1}{}^1$ and $c_{r_2}{}^1$ are congruent. The availability of the translations along the asymptotes to g allows a considerable strengthening of this statement. Each arc $\overset{\frown}{AB}$ on $L_1(P, \overleftarrow{g})$ is congruent to an arc on any $L_1(Q, \overleftarrow{g})$ starting at g and therefore also to an arc starting at an arbitrary point. (We say the arc $\overset{\frown}{AB}$ starts at A if A is on $L_1(P, \overleftarrow{g})$ between B and P.) To prove this we draw the asymptote a to $\overleftarrow{g}$ through B and the curve through A equidistant from a. Since $r(a, g) = 0$, this curve intersects g at a point A'. We draw $L(A', \overleftarrow{g})$ and put $a \cdot L(A', \overleftarrow{g}) = B'$. The curve $c_r{}^1$ through B' may intersect $L_1(Q, \overleftarrow{g})$ at Q'. We say that $\overset{\frown}{QQ'}$ is congruent to $\overset{\frown}{AB}$ (and write $\overset{\frown}{QQ'} \cong \overset{\frown}{AB}$). Since $(Q \to A')$ transforms $\overset{\frown}{QQ'}$ into $\overset{\frown}{A'B'}$, we have $\overset{\frown}{QQ'} \cong \overset{\frown}{A'B'}$. The translation $(B \to B')$ along a transforms $L(P, \overleftarrow{g})$ into the limit circle through B' to the image $\overleftarrow{g}'$ of $\overleftarrow{g}$ under $(B \to B')$. $\overleftarrow{g}'$ is an asymptote to $\overleftarrow{g}$ in π_2. The limit circles to $\overleftarrow{g}'$ and $\overleftarrow{g}$ coincide in π_1 on account of (1.13), hence $\overset{\frown}{QQ'} = \overset{\frown}{A'B'} = \overset{\frown}{AB}$. It follows from this that on each arc $\overset{\frown}{AB}$ of $L_1(P, \overleftarrow{g})$ there exists a point C (the center of AB) with $\overset{\frown}{AC} \cong \overset{\frown}{CB}$ and more generally points $C_1, \cdots, C_{n-1}$ with

$$\widehat{AC_1} \cong \widehat{C_1C_2} \cong \cdots \cong \widehat{C_{n-1}B}.$$

We then write

$$\widehat{PQ} = \widehat{A'B'} = 2\widehat{AC} = 2\widehat{CB}$$

$$\widehat{AB} = n\,\widehat{C_\nu C_{\nu+1}} = \frac{n}{m}\,\widehat{C_\nu C_{\nu+m}}, \text{ and so forth,}$$

and extend this notation to arbitrary non-negative real numbers by a limit process. A consideration similar to the proof of (2.5) shows that the limit circle $L_1(P, \overleftarrow{g})$ are convex and therefore rectifiable, and that congruent pieces have the same length. If one does not wish to use this fact, he may associate the number 1 to an arbitrary arc $\widehat{AB}$, $A \neq B$, of any $L_1(P, g)$ and real

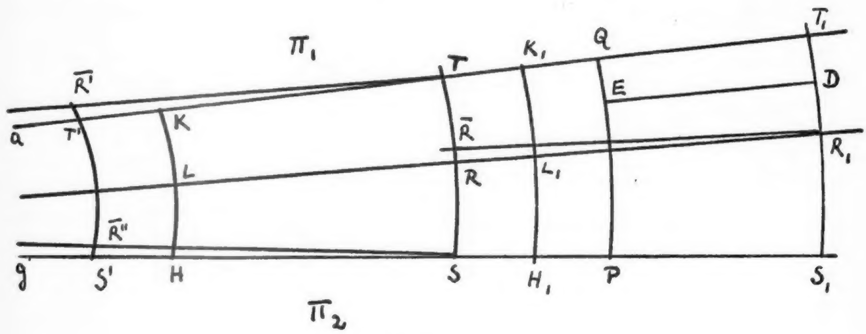

Fig. 1.

numbers to other arcs according to our above prescription. We use $\widehat{AB}$ both for the arc and its length.

We intend to determine the length of an arc $\widehat{YX}$, where X traverses g and Y a fixed asymptote a to $\overleftarrow{g}$ as a function of the distances on g (see Fig. 1). Let $\widehat{ST}$, $S \subset g$, $T \subset a$ be any such arc; draw the curve through T equidistant from g and the one through S equidistant from a. They may intersect in R_1. Let the limit circle through R_1 to g intersect g in S_1 and a in T_1. Then, as before, $\widehat{S_1R_1} \cong \widehat{R_1T_1}$. The asymptote to $\overleftarrow{g}$ through R_1 intersects $\widehat{ST}$ in its center $\bar{R}$. If the intersection were different from $\bar{R}$, say $R \subset S\bar{R} - \bar{R}$, then $(S_1 \to S)$ would transform $\widehat{S_1R_1}$ into $\widehat{ST}$, $\underline{R_1R}$ into a, and hence $\overrightarrow{R_1\bar{R}}$ into a ray $\overrightarrow{T\bar{R}'}$, where $\bar{R}'$ would be on the limit circle $L(S', \overrightarrow{g})$ through the image S' of S under $(S_1 \to S)$. Put $L(S', \overrightarrow{g}) = T'$. Then $(T_1 \to T)$ transforms $\widehat{T_1R_1}$

into $\widehat{TS}$, $\underline{R_1R}$ into g, $\widehat{TR}$ into $\widehat{T'S'}$, and $\overrightarrow{R_1\bar{R}}$ into a ray $\overrightarrow{S\bar{R}''}$, where $\bar{R}''$ is an interior point of $\widehat{T'S'}$. But then we should have

$$\widehat{T'\bar{R}''} \cong \widehat{T\bar{R}} \cong \widehat{\bar{R}S} \cong \widehat{\bar{R}'S'}$$

whereas $\widehat{T'\bar{R}''}$ is a proper subarc of $\widehat{\bar{R}'S'}$.

Since $\widehat{S_1R_1} = 2\widehat{RS}$, we see the s. l. connecting the centers of $\widehat{S_1R_1}$ and $\widehat{SR}$ is an asymptote to g and, since the arcs $\widehat{R_1T_1}$ and $\widehat{RT}$ also belong to limit circles to $\overrightarrow{R_1R}$, we see that the s. l. connecting the centers of $\widehat{R_1T_1}$ and $\widehat{RT}$ is also an asymptote to $\overleftarrow{g}$. We thus find:

Let the points $B_1 \subset \widehat{S_1T_1}$ and $A \subset \widehat{ST}$ be chosen in such manner that

$$\widehat{SA} = \lambda\widehat{ST}, \qquad \widehat{S_1A_1} = \lambda\widehat{S_1T_1}\ (= 2\lambda\widehat{ST}), \qquad 0 < \lambda < 1.$$

Then $\underline{AB}$ is an asymptote to g.

Furthermore our considerations show that

$$\widehat{S'T'} = \tfrac{1}{2}\widehat{ST} = \tfrac{1}{4}\widehat{S_1T_1}, \qquad r(S', S) = r(S, S_1).$$

More generally, if S_n and $S^{(n)}$ are chosen in such manner that

$$S_n \subset \overrightarrow{S'S} \text{ and } r(S, S_n) = nr(S', S) = nr(S, S_1)$$
$$S^{(n)} \subset \overrightarrow{SS'} \text{ and } r(S, S^{(n)}) = nr(S', S)$$

and if one puts $L(S^{(n)}, \overleftarrow{g}) \cdot a = T^{(n)}$, $L(S_n, \overleftarrow{g}) \cdot a = T_n$ one has

$$\widehat{S^{(n)}T^{(n)}} = \frac{1}{2^n}\widehat{ST}, \qquad \widehat{S_nT_n} = 2^n\widehat{ST}.$$

Now choose any point H between S' and S and put $a \cdot L(H, \overleftarrow{g}) = K$. Let H_1 be the point on g for which, with $L(H_1, \overleftarrow{g}) \cdot a = K_1$, one has $\widehat{H_1K_1} = 2\widehat{HK}$. We assert that

$$r(H, H_1) = r(S, S_1).$$

We consider the points H_n on $\overrightarrow{HH_1}$ with $r(H_n, H) = nr(H_1, H)$ and put $a \cdot L(H_n, \overleftarrow{g}) = K_n$. If, for instance, one had $r(H, H_1) > r(S, S_1)$ we should have

$$r(S, S_\nu) < r(S, H_\nu)$$

for large ν; hence with an obvious signification of the sign $>$

$$2^\nu \overset{\frown}{HK} = \overset{\frown}{H_\nu K_\nu} > \overset{\frown}{S_\nu T_\nu} = 2^\nu \overset{\frown}{ST}$$

in contradiction to the fact that $\overset{\frown}{ST} > \overset{\frown}{HK}$.

Up to this point we have found:

(3.2) *There exists a number $\sigma > 0$ such that, for an arbitrary asymptote b to $\overleftarrow{g}$ in π_1 and for an arbitrary pair of points $A < B$ on $\overleftarrow{g}$ with $r(A, B) = \sigma$, the arc of $L(A, \overleftarrow{g})$ between b and g is congruent to the half of the arc of $L(B, \overleftarrow{g})$ between g and b.*

We now prove:

(3.3) *The points Z of all arcs $\overset{\frown}{XY}$ of $L(X, \overleftarrow{g})$ with $X \subset g$, $Y \subset a$ and $\overset{\frown}{XZ} = \lambda XY$, $0 < \lambda < 1$, form an asymptote to g.*

As in a previous proof, it is sufficient to show this for $\lambda = \frac{1}{2}$. Using the same notations as before and calling L, L_1 the centers of $\overset{\frown}{HK}$, respectively $\overset{\frown}{H_1K_1}$, we conclude from $r(H, H_1) = r(S, S_1) = \sigma$ that $(H \rightarrow H_1) = (S \rightarrow S_1)$. But $(H \rightarrow H_1)$ transforms a into the asymptote to $\overleftarrow{g}$ through that point M of $\overset{\frown}{H_1K_1}$ for which $\overset{\frown}{H_1M} \cong \overset{\frown}{HK}$; hence $M = L_1$. $(S \rightarrow S_1)$ transforms a into $\underline{R_1R}$; hence $\underline{R_1R} \equiv \underline{L_1L}$. This proves the assertion.

Herewith the length of $\overset{\frown}{XY}$ can be determined as follows: Let P be the center of $\overline{SS_1}$ and put $L(P, \overleftarrow{g}) = Q$. $(S \rightarrow P)$ transforms $\overset{\frown}{ST}$ into a subarc $\overset{\frown}{PE}$ of $\overset{\frown}{PQ}$. Let the asymptote to $\overleftarrow{g}$ through E intersect $\overset{\frown}{S_1T_1}$ in D. Then $\overset{\frown}{PQ} \cong \overset{\frown}{S_1D}$ since $(S \rightarrow P)$ transforms a into $\underline{ED}$, and, on account of (3.3),

$$\frac{\overset{\frown}{ST}}{\overset{\frown}{PQ}} = \frac{\overset{\frown}{PE}}{\overset{\frown}{PQ}} = \frac{\overset{\frown}{S_1D}}{\overset{\frown}{S_1T_1}} = \frac{\overset{\frown}{PQ}}{2\overset{\frown}{ST}}$$

or

$$PQ = \sqrt{2}\, \overset{\frown}{ST}.$$

Putting

$$r(S,X) = x \text{ for } X \subset \overrightarrow{SS'}$$

$$r(S,X) = -x \text{ for } X \subset \overrightarrow{SS_1}$$

we find in this way

$$(3.4) \qquad \widehat{XY} = \widehat{ST}\, e^{-x[(\log 2)/\sigma]}$$

which (except for the normalization $\sigma = \log 2$) is the same formula as in hyperbolic geometry.[14]

We have derived (3.4) using only a part of the congruence axioms. We show, indeed, by an example that *a geometry with all translations along two non-parallel asymptotes is not necessarily hyperbolic, not even Desarguesian.*

We again introduce in the domain $x^2 + y^2 \leq 1$ of the (x,y)-plane the hyperbolic metric $h(P,Q)$ of —. Designate by h the broken line consisting of the two hyperbolic rays issuing from $(0,0)$ and ending at $(0,-1)$ respectively $(\sqrt{2}, \sqrt{2})$. We define distances for the points P', Q' on h as follows: Call g the hyperbolic s. l. $y = 0$, $|x| < 1$ and let c_r^1, c_r^2 be the hyperbolic curves equidistant from g in $y < 0$ respectively $y > 0$. Each of these curves c_r^1, c_r^2 intersects h exactly once. We draw the curves passing through P' and Q'. If both of them are curves c_r^1 (c_r^2), say $c_{r_1}^1, c_{r_2}^1$ ($c_{r_1}^2, c_{r_2}^2$), we put $r(P',Q') = |r_1 - r_2|$. If one is the curve $c_{r_1}^1$, the other $c_{r_2}^2$ we put $r(P',Q') = r_1 + r_2$. Call $\bar{G}$ the smallest group of hyperbolic motions containing the translations along the s. l. ending at $(-1,0)$. $\bar{G}$ can be generated by the translations along g and the hyperbolic rotations around the point $(-1,0)$. By a suitable transformation of $\bar{G}$ any pair of points P, Q, such that $\underline{PQ}$ (by $\underline{XY}$, respectively $\overline{XY}$, we designate during the discussion of this example the Euclidean straight line, respectively segment, connecting X and Y) does not pass through $(-1,0)$, can be transformed into exactly one pair of points P', Q' on h. We put

$$(3.5\text{a}) \qquad r(P,Q) = r(P',Q').$$

For pairs of points P, Q, where $\underline{PQ}$ passes through $(-1,0)$ we put

$$(3.5\text{b}) \qquad r(P,Q) = h(P,Q).$$

It is easy, but a little tedious, to confirm that the function $r(P,Q)$ defines an SL space. The s. l. of this metric arc h and its transforms under $\bar{G}$ and the hyperbolic s. l. issuing from $(-1,0)$. Evidently $r(P,Q)$ remains in-

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[14] See for instance [6], p. 55.

variant under all transformations of G. The theorem of Desargues does not hold.

To find a configuration confirming this statement (comp. Fig. 2), we take the Euclidean ellipse e with center at $(-\frac{1}{4}, 0)$ having third order contact with the unit circle at $(-1, 0)$. e is a limit circle in both our and the hyper-

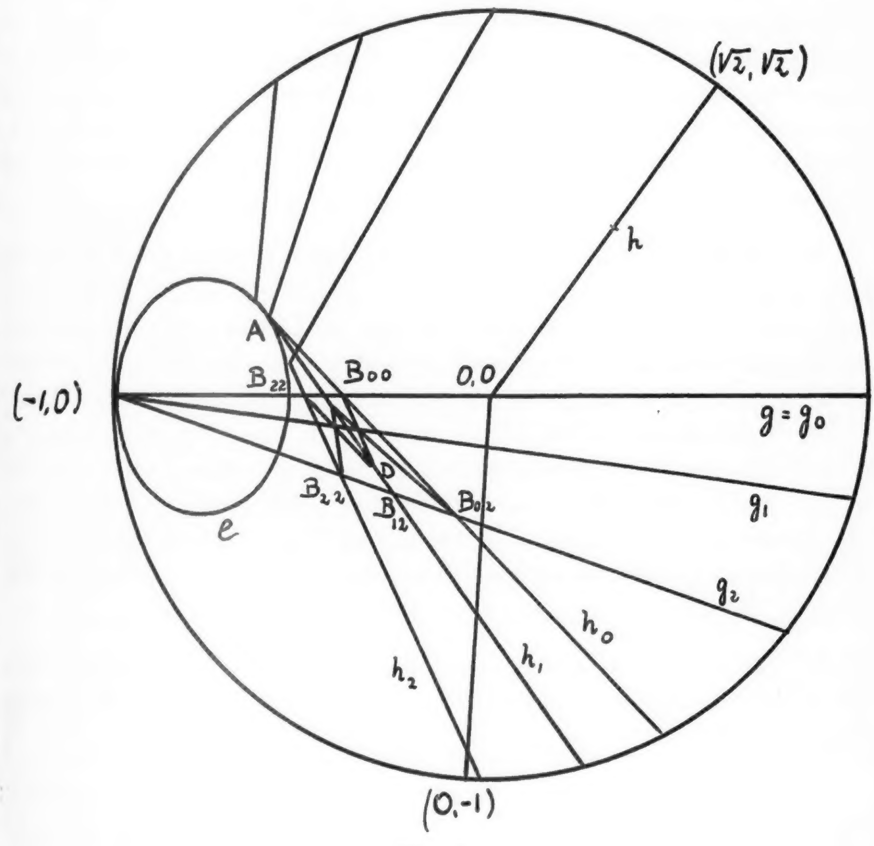

Fig. 2.

bolic metric:[15] Take a point A on e in $y > 0$ such that the hyperbolic tangent to e at A intersects g at a point B_{10}. A divides this tangent into two rays of which the one containing B_{10} is a part of an s. l., h_1, of our metric. Choose g_2 issuing from $(-1, 0)$ such that it intersects h_1 in a point B_{12}. Take now two hyperbolic rays issuing from A and enclosing a sufficiently small angle α which contains h_1 in its interior. Let these rays intersect g_i in B_{0i} and B_{2i}, $(i = 0, 2;$

[15] See [13], p. 356.

$g_0 = g$) where B_{2i} is to be understood to be between $(-1, 0)$ and B_{0i}. Then the hyperbolic rays $\overrightarrow{AB_{00}}$ and $\overrightarrow{B_{20}B_{22}}$ are also rays of our metric. Let h_0 and h_2 be the s. l. of our metric containing $\overrightarrow{AB_{00}}$ and $\overrightarrow{B_{20}B_{22}}$, respectively. Then h_0 and h_1 pass through A but h_2 does not. We choose D and E on $\overline{B_{10}B_{12}}$ $(h_1 g_i = B_{1i})$ such that D is between E and B_{12}. If α is sufficiently small, the hyperbolic segments $\overline{B_{00}D}$, $\overline{B_{20}D}$, $\overline{B_{02}E}$ and $\overline{B_{22}E}$ are also segments of our metric. The points $\overline{B_{00}D} \cdot \overline{B_{02}E}$ and $\overline{B_{20}D} \cdot \overline{B_{22}E}$ together with $(-1, 0)$ lie on a hyperbolic s. l. g_1, which is also an s. l. of our metric. Hence corresponding sides of the triangles $B_{00}DB_{20}$ and $B_{02}EB_{22}$ intersect on g_1, but the s. l. h_0, h_1, and h_2 through corresponding vertices of these triangles do not pass through one point.

4. Geometries with translations along two s. l. which are not asymptotes to each other. We now assume that all translations along two s. l. g and h exist where h is not an asymptote to g (and hence g not to h, compare (3. 1)). If h and g do not intersect, the curve which touches h and is equidistant from g intersects the curve which touches g and is equidistant from h in exactly two points. (This follows from (2. 6), (2. 7) and page 242). If A is one of these points, a suitable translation along g puts h into a position so that it passes through A and a translation along h does the same for g. These two s. l. through A are different and all translations along them exist. We see that we can restrict ourselves to the case where h and g have a common point 0. We are going to prove the following theorem.

(4. 1) *If all translations along g and h exist, if h is not an asymptote to g, and if there is either a parallel $g' \neq g$ to g or a parallel $h' \neq h$ to h, then the metric is Minkowskian.*

Let a parallel $g' \neq g$ exist. We assume that h intersects g in a point 0; then it must also intersect g' in a point $0'$. $(0 \to 0')$ transforms g into an s. l. not intersecting g (comp. p. 239) through $0'$, hence into g'; and transforms g' into an s. l. g'' intersecting h in the image $0''$ of $0'$ under $(0 \to 0')$. Since g' is equidistant from g, g'' must be equidistant from g'; hence parallel to g'. By continuing this and by considering also the images of g' under $(0' \to 0)$, $(0' \to 0)^2, \cdots$ we find, with the help of (2. 7) and (2. 10), that the parallel axiom holds with respect to g, and that these parallels are equidistant from each other. Furthermore, the images of h under any two translations along g cut on h equal segments of all these parallels to g, and any two parallels to g cut out equal segments of all the images of h. If we knew that the parallel

axiom holds with respect to each s. l. in Σ we could derive our theorems from these statements in few lines. The length of the following proof is due to the fact that we prove the parallel axiom and the theorem of Desargues simultaneously.

We introduce coördinates in our plane Σ. We distinguish the two rays g^+ and g^- of g issuing from 0 and the two rays h^+ and h^- of h issuing from 0. Through each point P of Σ there passes exactly one image h_x of h under a translation along h and exactly one parallel g_y to g. Put $h \cdot g_y = Y$ and $g \cdot h_x = X$. We have, as stated before,

$$r(X, 0) = r(P, Y), \qquad r(Y, 0) = r(P, X).$$

As coördinates X, Y of P we take

$$\begin{array}{ll} x = r(X, 0) \text{ if } X \subset g^+, & x = -r(X, 0) \text{ if } X \subset g^- \\ y = r(Y, 0) \text{ if } Y \subset h^+, & y = -r(Y, 0) \text{ if } Y \subset h^-. \end{array}$$

We map Σ onto a Cartesian (x, y)-plane $\bar{\Sigma}$ by making points with the same coördinates correspond. To a motion of Σ composed of translations along g and h corresponds a translation of $\bar{\Sigma}$, and, since each translation in $\bar{\Sigma}$ can be composed of translations along the x- and the y-axes, there corresponds a a motion of Σ to each translation of $\bar{\Sigma}$.

We wish to show that each s. l. in Σ is mapped onto a straight line in $\bar{\Sigma}$. We first remark that, if any s. l. k in Σ is mapped onto a straight line $\bar{k}$ in $\bar{\Sigma}$, then all translations along k exist because all translations of $\bar{\Sigma}$ along $\bar{k}$ exist. Let k' be any transform of k by a translation along g. Since k and k' cut out equal segments of g and its parallels and a translation along k' transforms a parallel to g into a parallel to g, we see that under the translation along k the s. l. k' is transformed into itself. It is therefore equidistant from k and parallel to k. We have

(4.2) *If any* s. l. k *in* Σ *is mapped onto a straight line* $\bar{k}$ *in* $\bar{\Sigma}$, *then the parallel axiom holds with respect to* k. *The images of the parallels to* k *in* Σ *are the Euclidean parallels to* $\bar{k}$ *in* $\bar{\Sigma}$. *All translations along* k *exist.*

Let e be any s. l. in Σ, $\bar{e}$ its image in $\bar{\Sigma}$. To each curve in $\bar{\Sigma}$ which can be deduced from $\bar{e}$ by a translation there corresponds an s. l. in Σ, since e can be carried into this s. l. by a motion of Σ. Therefore we know that by any translation of $\bar{\Sigma}$ the curve $\bar{e}$ is transformed into a curve which either coincides with $\bar{e}$, or intersects $\bar{e}$ in one point, or is disjoined from $\bar{e}$. If $\bar{e}$ has no intersections with one of its images without being identical with it, $\bar{e}$ is a straight

line. If $\bar{e}$ has intersections with some of its images, the fact that it has at most one intersection with each of them, implies that $\bar{e}$ is a convex curve without parallel supporting lines.[16] Therefore $\bar{e}$ has two different, well defined asymptotes (in the Euclidean sense).

Now let us assume that there exists an s. l. e_0 through 0 in Σ whose image $\bar{e}_0$ is a convex curve but not a straight line. Let e_ϵ be an s. l. through 0 tending to e_0. Then the image $\bar{e}_\epsilon$ of e_ϵ tends to $\bar{e}_0$. The angle over which the directions of the supporting lines to $\bar{e}_\epsilon$ range tends to an angle containing the corresponding angle for $\bar{e}_0$.[17] Since we assume that the latter is positive, for small ϵ the former one is positive too and has a common interior direction with the latter. Then one can find chords $\bar{\sigma}_0$ and $\bar{\sigma}_\epsilon$ of $\bar{e}_0$ respectively $\bar{e}_\epsilon$ which are parallel to the common direction and have the same length (one or both of these chords may be subarcs of $\bar{e}_0$ respectively $\bar{e}_\epsilon$). By a translation of $\bar{\Sigma}$ the chord $\bar{\sigma}_\epsilon$ can be carried into $\bar{\sigma}_0$. Let $\bar{e}'$ be the image of $\bar{e}_\epsilon$ under this translation. $\bar{e}'$ and $\bar{e}_0$ have two common points; hence (since $\bar{e}'$ is the image of an s. l. e' in Σ) $\bar{e}' \equiv \bar{e}_0$. We see, then, that

(4. 3) *For small ϵ, the curve $\bar{e}_\epsilon$ can be transformed into $\bar{e}_0$ by a translation of $\bar{\Sigma}$.*

We now consider all s. l. through 0 in Σ. The images of some of them are straight lines. These straight lines form a closed set. We consider one, $\bar{W}$, of the open angles which is bounded by two such straight lines $\bar{e}_1$, $\bar{e}_2$, but contains none of them. From (4. 3) it follows that all the s. l. e of Σ in W are mapped onto convex curves $\bar{e}$ in $\bar{W}$, each of which can be transformed into any other by a translation of $\bar{\Sigma}$. If $e \rightarrow e_1$ ($e \rightarrow e_2$), we must have $\bar{e} \rightarrow \bar{e}_1$ ($\bar{e} \rightarrow \bar{e}_2$). Therefore, one of the Euclidean asymptotes, $\bar{e}'_1$, of $\bar{e}$ must be parallel to $\bar{e}_1$ and the other must be parallel to $\bar{e}_2$. Since the Euclidean distance of $\bar{e}'_1$ and $\bar{e}$ vanishes, one sees from the fact that translations of $\bar{\Sigma}$ correspond to motions of Σ, that $r(e'_1, e) = 0$, also. According to (4. 2) e'_1 is an s. l. and on account of (1. 16), e is an asymptote to e'_1 such that e_1 and e would be two different asymptotes to e'_1 through 0. We have proved that all s. l. through 0 are mapped onto straight lines through $\bar{0}$. With the help of (4. 2) and (2. 10) one concludes easily that the metric is Minkowskian.

The other case, where there exists neither a parallel to g nor to h is much easier to deal with. We prove

[16] This can easily be shown. One finds a proof for this and other questions connected with it in [10].

[17] A proof for the analogous fact in a space of any dimension can be found in [1], p. 35.

(4.4) *If all translations along g and h exist, where h is not an asymptote to g, and if through each point of Σ not on g there pass two different asymptotes to g and through each point not on h two different asymptotes to h, then the metric is hyperbolic.*

The idea of the proof is obvious: One considers two non-intersecting s. l. g_{x_1} and g_{x_2} along which all translations exist and which are not asymptotes to each other. Then one takes a point S between these s. l. and the curves equidistant from g_{x_1} and g_{x_2} through S. They will (with a proper choice of S) intersect in a further point S'. The translation along g_{x_1} carrying S into S' followed by that along g_{x_2} carrying S' into S is a rotation around S. By keeping S fixed and varying g_{x_1} continuously, one varies the rotations around S continuously and gets in this way the full group of rotations around S. To make this a rigorous proof one has to show that g_{x_1} belongs to a continuous family of s. l. which are not asymptotes to g_{x_2} and along which all translations exist, and, furthermore, that the rotations around S really vary when g_{x_1} moves within that family.

To carry this out in detail, we prove first, that, if the s. l. g and h (of 4.4) intersect (which according to p. 250 always can be supposed to be the case), the transforms of g under the translations along h are not asymptotes to each other.

Put $g \cdot h = 0$. If by $(0 \rightarrow 0')$, $0' \subset h$, g were transformed into the asymptote g' to $\overleftarrow{g}$ through $0'$, it would easily follow from p. 242 that each translation along h would transform g into an asymptote to $\overleftarrow{g}$. Therefore we would have the situation of section 3. $L(0, \overleftarrow{g})$ does not coincide with h; otherwise, a translation $(0 \rightarrow P')$ along g would transform $L(0, \overleftarrow{g})$ into $L(P', \overleftarrow{g})$ and, on the other hand, into an s. l. h'. Since $L(0, \overleftarrow{g})$ and $L(P', \overleftarrow{g})$ are equidistant, h and h' would be equidistant, and therefore parallel. Because $L(0, \overleftarrow{g})$ is different from h, a suitable asymptote g_1 to $\overleftarrow{g}$ would intersect $L(0, g)$ in a point Q_{11} such that the segment $\overline{Q_{11}Q_{10}}$ of g_1 between $L(0, \overleftarrow{g})$ and h has a length of the form $\sigma/2^n$, where σ has the same significance as in (3.2). In order to fix the ideas, let this segment be in the exterior of $L(0, \overleftarrow{g})$. Let the curve c_1 equidistant from h and through Q_{11} intersect g in $Q_{0,1}$. Let us introduce the point $Q_{0,2}$ on g, for which $Q_{0,1}$ is the center of $\overline{Q_{0,2}0}$, the curve c_2 equidistant from h and through $Q_{0,2}$, and the points $Q_{i,0}$ on $\overrightarrow{0Q_{1,0}}$ such that

$$r(Q_{i,0}, 0) = i \cdot r(Q_{1,0}, 0).$$

The asymptote to g through $Q_{i,0}$ may intersect c_1 in $Q_{i,1}$ and c_2 in $Q_{i,2}$. Then it follows from our assumption (that the asymptotes to g are the images of g under the translations along h) that all segments $\overline{Q_{i,j}\, Q_{i,j+1}}$ have length $\sigma/2^n$; that $L(0, \overleftarrow{g})$ passes through Q_{11} and Q_{22}; and that the limit circle to $\overleftarrow{g}$ through $Q_{i,0}$ passes through the points $Q_{i+1,1}$ and $Q_{i+2,2}$. From the signification of σ it follows, then, that $(Q_{0,2} \to 0)$ carries $Q_{2,2}$ into $Q_{1,0}$, $Q_{3,2}$ into $Q_{2,0}$ and $Q_{4,2}$

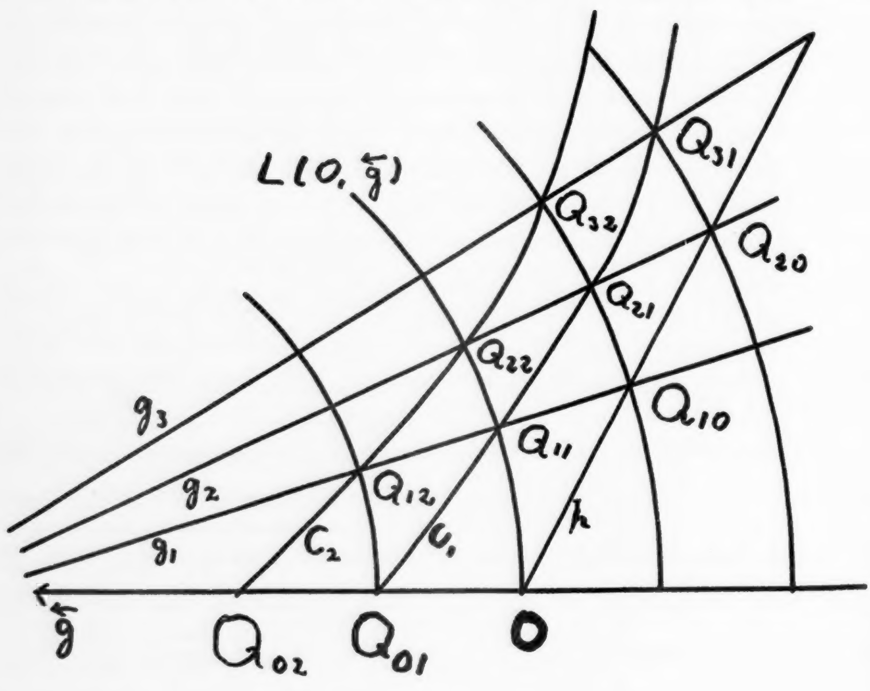

Fig. 3.

into $Q_{3,0}$. By considering $(0 \to Q_{0,2})$, one sees that $Q_{1,2}$, $Q_{2,2}$, $Q_{3,2}$ must lie on one s. l.; hence it would follow from (2. 5) to (2. 7) that the curve c_1 must be an s. l. parallel to h, in contradiction to the assumptions of (4. 4).

We designate the transforms of g under $(0 \to X)$, $X \subset h$, by g_x (see Fig. 4). Let X_0, X_1, X_2 be three different points on h such that X_0 is between X_1 and X_2. We choose a point S on g_{x_0} so that the curves c_1 equidistant from g_{x_1} and c_2 equidistant from g_{x_2}, through S, both cross g_{x_0} and contain in their exteriors, except for S, the same one of the two rays of g_{x_0} determined by S. This is always possible if X_0 is sufficiently near to X_1. Let S_{x_1} be the other

intersection of c_1 and c_2. We consider the translations T_1 and T_2 along g_{x_1} respectively g_{x_2} which carry S into S_{x_1}. If S_{x_1} is between g_{x_0} and g_{x_2} we form $T_2T_1^{-1}$ (first T_2 then T_1^{-1}); if S_{x_1} is between g_{x_0} and g_{x_1} we form $T_1T_2^{-1}$ (for $S_{x_1} \subset g_{x_0}$ either of these motions will do). Assume the first case and put $ST_2^{-1} = K$. By $T_2T_1^{-1}$ the point K is carried into a point K_{x_1} on the side of g_{x_0} other than the one containing K. $T_2T_1^{-1}$ is a rotation with center S different from the identity. We now make the analogous transformation with g_x instead of g_{x_1}, $X \subset \overline{X_0X_1}$: We draw the curve equidistant from g_x and

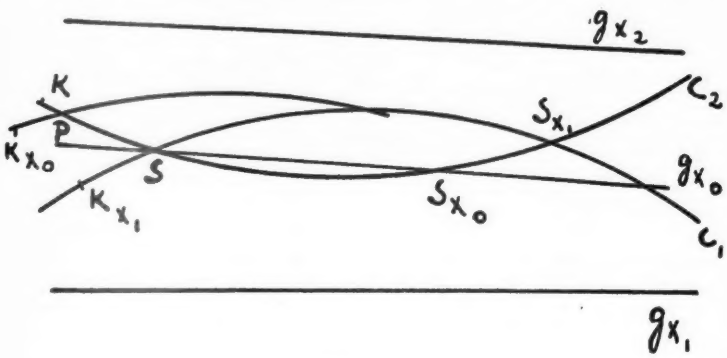

Fig. 4.

through S which may intersect c_2 in S_x (in addition to S). The motion of Σ composed of the translation along g_{x_2} carrying S into S_x and the translation along g_x carrying S_x into S may move K into K_x. The points K_x are on the circle $K(S,K)$; K_x depends in a continuous manner on g_x and does not remain fixed if X varies from X_1 to X_0. For, if $g_x = g_{x_0}$ $(= c_{x_0})$, the point S_{x_0} is on c_2 between S and S_{x_1}; therefore, the translation along g_{x_2} moving S into S_{x_0} moves K into a point P on c_2 between K and S, and $(S_{x_0} \to S)$ moves P into K_{x_0} which, therefore, is between g_{x_0} and g_{x_2} (or on g_{x_0}, if g_{x_0} passes through S and S_{x_1}) and different from K_{x_1}.

Thus there exist rotations with S as center transforming K into all points of an arc of $K(S,K)$ leading from K_{x_1} to K_{x_0}; hence, the full group of rotations around S exists. Since, by suitable translations along g and h the point S can be moved into each point of the plane, all rotations around each point exist. It is well known that this implies that the metric is hyperbolic.[18]

[18] The only place where this fact is stated exactly in the form used here, seems to be [5], Anhang IV. Of course this is much too deep a result for our purpose; one verifies easily in a direct way that the axioms of the hyperbolic plane geometry hold (in any of the current forms).

The theorems (4.1) and (4.4) and the examples (2.11) and (3.5) give together

If there exist all translations along two shortest lines which are neither parallel nor asymptotes to each other, then the metric is Desarguesian, namely, either Minkowskian or hyperbolic. If h and g are asymptotes or parallel to each other the metric is not necessarily Desarguesian, even if one assumes in addition that the parallel axiom either in the Euclidean or in the hyperbolic form holds with respect to each shortest line.

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